4U TRIAL 2002

Question 1

a) $\int \frac{dx}{x \log x} = \log(\log x) + c$

b) $\int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1}$

$= \tan^{-1}(x+3) + c$

c) $u = \sqrt{1-x}$ when $x = 0$, $u = 1$

$x = 1-u^2$ and $x = 1$, $u = 0$

and $dx = -2u\,du$

$$\int_0^1 x^2\sqrt{1-x}\,dx = \int_1^0 (1-u^2)^2 . u . -2u\,du$$

$$= \int_0^1 2u^2(1-u^2)^2\,du$$

$$= \int_0^1 2u^2 - 4u^4 + 2u^6\,du$$

$$= \left[\frac{2u^3}{3} - \frac{4u^5}{5} + \frac{2u^7}{7}\right]_0^1$$

$$= \frac{16}{105}$$

d) $\int_0^1 \sin^{-1}x\,dx =$

let $u = \sin^{-1}x$ $\quad v = x$

$du = \frac{1}{\sqrt{1-x^2}}$ $\quad dv = 1$

$$= \left[x\sin^{-1}x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} + \frac{1}{\cancel{2}}\left[\cancel{2}(1-x^2)^{\frac{1}{2}}\right]_0^1$$

$$= \frac{\pi}{2} + [0 - 1]$$

$$= \frac{\pi}{2} - 1$$

e) $5x^2 - 5x + 14 = (ax+b)(x-2) + c(x^2+4)$

$= ax^2 + (-2a+b)x - 2b + cx^2 + 4c$

$= (a+c)x^2 - (2a+b)x - 2b + 4c$

$$\left.\begin{array}{l} a+c = 5 \\ -2a+b = -5 \\ -2b+4c = 14 \end{array}\right\} \begin{array}{l} \text{—— } 1 \\ \text{—— } 2 \\ \text{—— } 3 \end{array}$$

From 1; $c = (5-a)$ and ③ becomes $-2b + 4(5-a) = 14$

$-b + 10 - 2a = 7$

$2a + b = 3$

$b = (3-2a)$.

substitute into ②: $-2a + (3-2a) = -5$

$\therefore a = 2,\ b = 1,\ c = 3$

(ii) $$\int \frac{5x^2 - 5x + 14}{(x^2+4)(x-2)}\, dx = \int \frac{2x+1}{x^2+4} + \frac{3}{x-2}\, dx$$

$$= \int \frac{2x}{x^2+4} + \frac{1}{x^2+4} + \frac{3}{x-2}\, dx$$

$$= \ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + 3\ln(x-2) + c$$

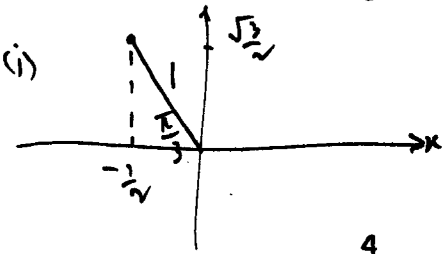
Question 2

a) $\dfrac{1}{zw} = \dfrac{1}{(1+2i)(3+i)}$

$= \dfrac{1}{1+7i} \times \dfrac{1-7i}{1-7i}$

$= \dfrac{1-7i}{50}$

b) (i)



$\frac{1}{2}(-1+i\sqrt{3}) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

(ii) $\frac{1}{16}(-1+i\sqrt{3})^4 = \left\{\frac{1}{2}(-1+i\sqrt{3})\right\}^4$

Hence

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^4$

$= \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= \frac{1}{2}(-1+i\sqrt{3})$

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or

$\frac{1}{16}(-1+i\sqrt{3})^4 = \frac{1}{16}\left((-1)^4 + 4(-1)^3 i\sqrt{3} + 6(-1)^2(i\sqrt{3})^2 + 4(-1)(i\sqrt{3})^3 + (i\sqrt{3})^4\right)$

otherwise

$= \frac{1}{16}\left(1 - 4i\sqrt{3},\ -6\times3 + 4i.3\sqrt{3} + 9\right)$

$= \frac{1}{16}\left(-8 + 8i\sqrt{3}\right)$

$= \frac{8}{16}(-1+i\sqrt{3})$

$= \frac{1}{2}(-1+i\sqrt{3})$

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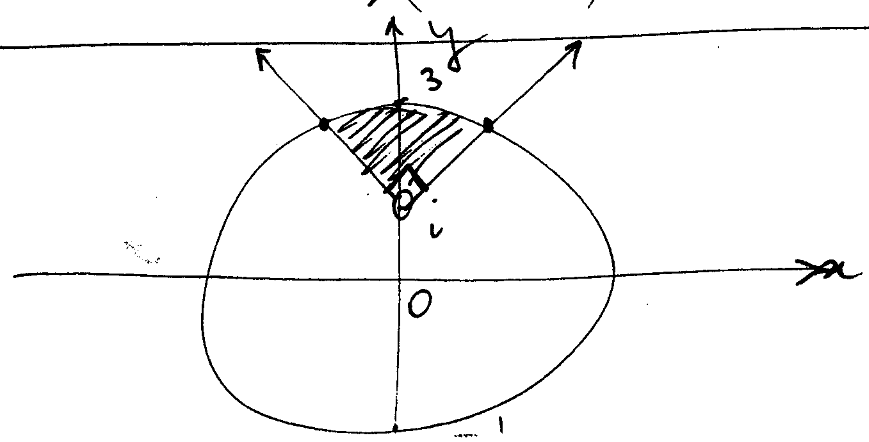
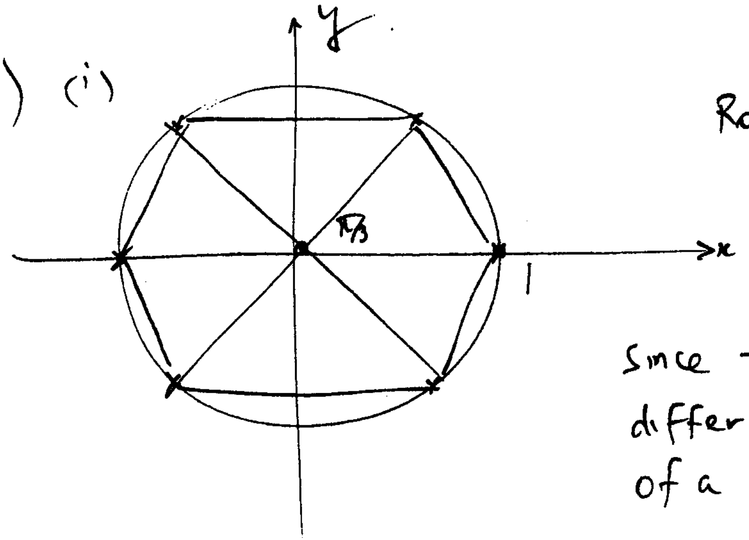
c)



d) (i) $|z| = 1$

(ii) $|z| = 1$ and $\arg z = \pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$.

e) (i)



Roots are ± 1, $\text{cis}\,\frac{\pi}{3}$, $\text{cis}\,\frac{2\pi}{3}$, $\text{cis}\,\frac{4\pi}{3}$, $\text{cis}\,\frac{5\pi}{3}$

(or $\text{cis}\,\frac{k\pi}{3}$ for $k = 0, 1 \ldots 5$)

Since their moduli equal 1, their arguments differ by $\frac{\pi}{3}$ they form the vertices of a regular hexagon on the unit circle.

(ii) $z^6 - 1 = (z^3)^2 - 1$

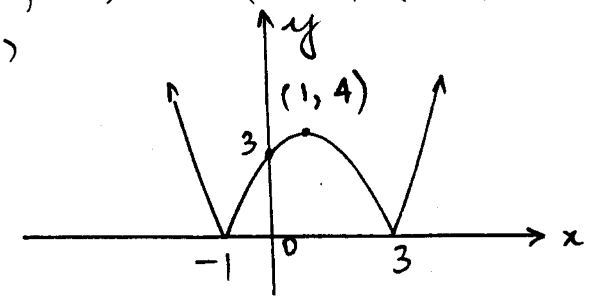
$= (z^3 + 1)(z^3 - 1)$

$= (z+1)(z^2 - z + 1)(z-1)(z^2 + z + 1)$

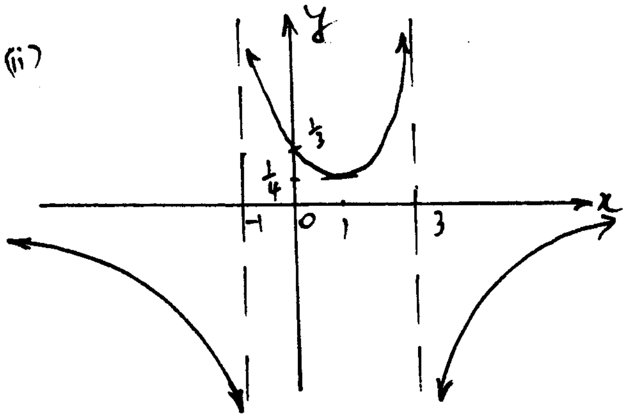
QUESTION 3

a) $f(x) = -(x-3)(x+1)$

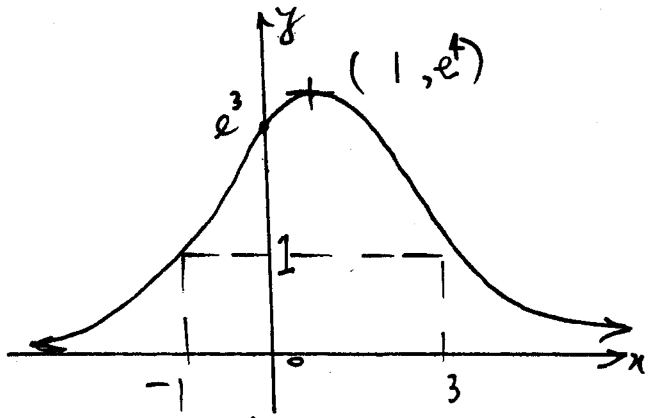
(i)



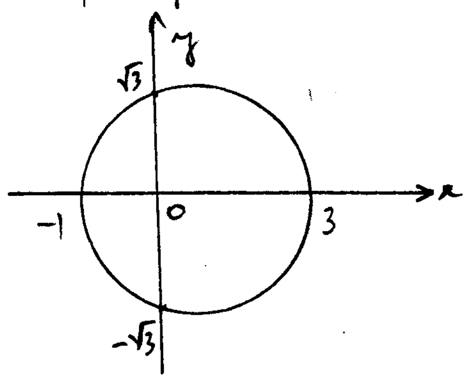
(ii)



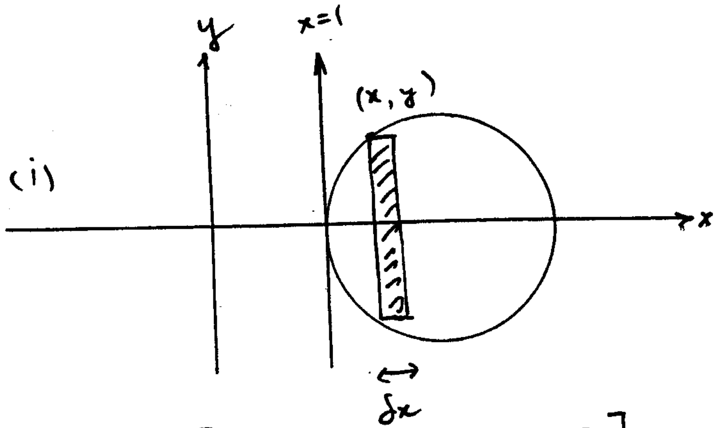
(iii)



(iv)



b) (i)



$$\delta V = \pi\left[(x+\delta x-1)^2-(x-1)^2\right]2y$$

$$= 2\pi y\left[(x+\delta x-1+x-1)(x+\delta x-1-x+1)\right]$$

$$= 2\pi y\left(2(x-1)+\delta x\right)\delta x$$

$$= 4\pi(x-1)y\,\delta x \qquad (\text{since } \delta x^2 \text{ terms can be ignored})$$

$$= 4\pi(x-1)\sqrt{1-(x-2)^2}\,\delta x$$

$$V = \lim_{\delta x\to 0} 4\pi\sum_{x=1}^{3}(x-1)\sqrt{1-(x-2)^2}\,\delta x$$

$$= 4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$$

(ii) Let $(x-2) = \sin\theta$ then $dx = \cos\theta\,d\theta$.

When $x = 1$, $\sin\theta = -1$ $\therefore \theta = -\frac{\pi}{2}$ and $x = 3$, $\sin\theta = 1$ $\theta = \frac{\pi}{2}$

So $$V = 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin\theta)\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

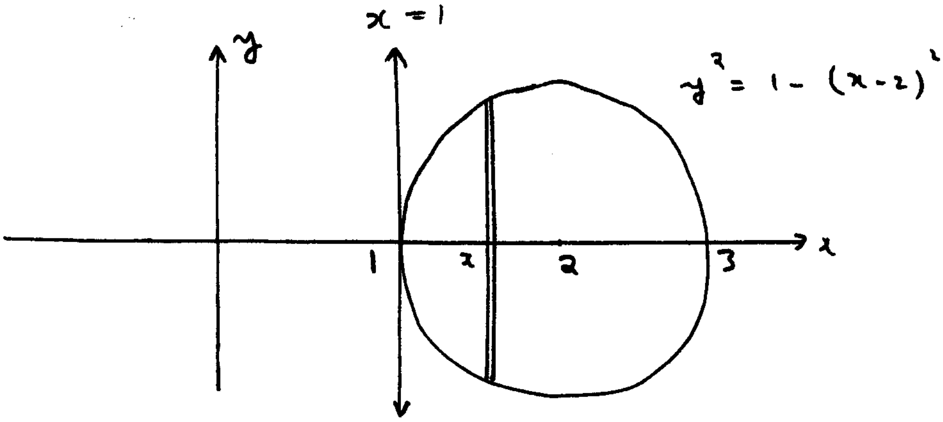
$$= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\sin\theta)\cos^2\theta\,d\theta = 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{1}{2}(1+\cos 2\theta)+\sin\theta\cos^2\theta\,d\theta$$

$$= 4\pi\left[\tfrac{1}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)-\tfrac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi\left[\frac{\pi}{2}+0-0\right]$$

$$= 2\pi^2$$

Q3(b) Alternative Approach



(i) Slice the region perpendicular to the x-axis, as shown. When the strip sketched above is rotated about the line $x = 1$, it generates a cylindrical shell with radius $x - 1$ and height $2y$.

Thus surface area of shell $= 2\pi(x-1) \times 2y$

$$= 4\pi(x-1)\sqrt{1-(x-2)^2}.$$

Hence volume of solid $= \int_1^3 4\pi(x-1)\sqrt{1-(x-2)^2}\,dx.$

(ii) Let $u = x - 2$.

Then $du = dx$

When $x = 1$, $u = -1$

When $x = 3$, $u = 1$

So volume $= 4\pi \int_{-1}^{1} (u+1)\sqrt{1-u^2}\,du$

$$= 4\pi \int_{-1}^{1} \sqrt{1-u^2}\,du + 4\pi \int_{-1}^{1} u\sqrt{1-u^2}\,du$$

The first integral $= 4\pi \times$ (semicircle of radius 1)

$$= 4\pi \times \frac{\pi}{2}$$

$$= 2\pi^2$$

The second integrand is odd, so the integral is zero.

Hence volume $= 2\pi^2$ cubic units.

Question 4

(a) Since $P(x)$ has a double root, then $P(x)$ and $P'(x)$ share a root.

$P(x) = 12x^3 + 44x^2 - 5x - 100$

$P'(x) = 36x^2 + 88x - 5$

$= (18x-1)(2x+5)$

So $P'(x)$ has roots $\frac{1}{18}$ and $\frac{-5}{2}$.

Now $P(\frac{1}{18}) \neq 0$ and $P(-\frac{5}{2}) = 0$, so $x = -\frac{5}{2}$ is the double root.

Let β be the other root, then $P(x) = k(x+\frac{5}{2})^2(x-\beta)$.

ie/ $k(x+\frac{5}{2})^2(x-\beta) = 12x^3 + 44x^2 - 5x - 100$

$\therefore k = 12$ (comparing coefficients.

and $12(+\frac{5}{2})^2(-\beta) = -100$

$\beta = \frac{100}{12} \times \frac{4}{25}$

$= \frac{4}{3}$.

So the roots are $-\frac{5}{2}, -\frac{5}{2}, \frac{4}{3}$.

b) (i) False, since $e^{-\frac{1}{2}x^2} >$ for all x

(ii) True because $\tan x$ and hence $\tan^7 x$ is odd

(iii) False – the integral is zero because $\cos x$, and hence $\cos 9x$, has point symmetry in the interval, so the integral is zero.

(iv) For $0 < x < 1$,

$0 < x^8 < x^7 < 1$

$1 < 1+x^8 < 1+x^7 < 2$

$1 < \sqrt{1+x^8} < \sqrt{1+x^7} < \sqrt{2}$

$1 > \frac{1}{\sqrt{1+x^8}} > \frac{1}{\sqrt{1+x^7}} > \frac{1}{\sqrt{2}}$

So $1 > \int_0^1 \frac{dx}{\sqrt{1+x^8}} > \int_0^1 \frac{dx}{\sqrt{1+x^7}} > \frac{1}{\sqrt{2}}$ and the statement is false.

(c) (i) $\angle APC = \angle AMC = 90°$ and are subtended by the chord AC and so these are angles on the circumference of a circle. Thus APMC are concyclic

(ii) $\angle PMA = \angle PCA = \theta$ (angles subtended by the chord AP at the circumference are equal)

(iii) $\angle PAM = \alpha = \angle PCM$ (angles at circumference)

$\angle PBC = \theta$ (base angles of ΔABC are equal)

$\therefore \Delta MPA \parallel\!\parallel \Delta BPC$ (A.A.)

(iv) $\dfrac{PA}{PC} = \tan\theta$ and $\dfrac{MA}{BC} = \dfrac{4}{6} = \dfrac{2}{3}$ (MA = 4 from Pythagoras' theorem)

$\dfrac{PA}{PC} = \dfrac{MA}{BC}$ (ratio of corresponding sides in similar Δ)

ie, $\tan\theta = \dfrac{2}{3}$.

(v) $\dfrac{DC}{\sin\theta} = \dfrac{BC}{\sin(\angle BDC)}$

$$DC = \frac{6\sin\theta}{\sin(180 - (\theta + \angle DCB))}$$

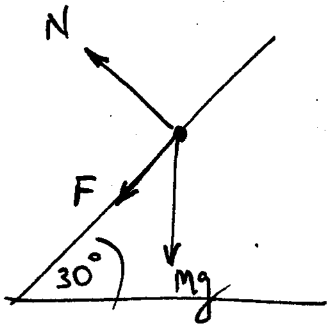
$$= \frac{6\sin\theta}{\sin(\theta + \angle DCB)}$$

$$= \frac{6\sin\theta}{\sin\theta\cos\angle DCB + \cos\theta\sin\angle DCB}$$

$$= \frac{6}{\cos\angle DCB + \sin\angle DCB \cdot \cot\theta}$$

$$= \frac{6}{\frac{3}{5} + \left(\frac{3}{2}\right)\left(\frac{4}{5}\right)} = \frac{10}{3}.$$

Question 5

a) 

$F = \frac{1}{10} N$

Vertically : $N \cos 30 - F \sin 30 = Mg$

$10F \frac{\sqrt{3}}{2} - \frac{F}{2} = Mg$

$F \left(\frac{10\sqrt{3}-1}{2} \right) = Mg$ ——— ①

Horizontally: $N \sin 30 + F \cos 30 = \frac{Mv^2}{r}$

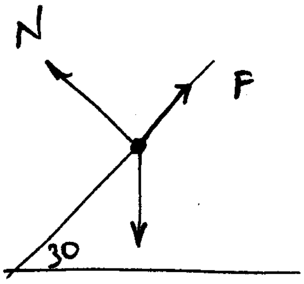
$10 \times \frac{F}{2} + F \times \frac{\sqrt{3}}{2} = \frac{Mv^2}{20}$

$20F\left(5 + \frac{\sqrt{3}}{2}\right) = Mv^2$ ——— ②

Dividing ② by ①: $v^2 = \dfrac{20Fg\left(5 + \frac{\sqrt{3}}{2}\right)}{\frac{F}{2}\left(10\sqrt{3}-1\right)}$

$= \dfrac{20g(10+\sqrt{3})}{(10\sqrt{3}-1)}$

So $v \leq \sqrt{\dfrac{20g\,(10+\sqrt{3})}{(10\sqrt{3}-1)}}$



Vertically : $N \cos 30 + F \sin \theta = Mg$

$10F\frac{\sqrt{3}}{2} + \frac{F}{2} = Mg$

$\frac{F}{2}\left(10\sqrt{3}+1\right) = Mg$ ——— ①

Horizontally: $N \sin 30 - F\cos 30 = \frac{Mv^2}{r}$

$10F \times \frac{1}{2} - F \times \frac{\sqrt{3}}{2} = \frac{Mv^2}{20}$

$\frac{\cancel{20}^{10}F}{\cancel{2}}\left(10 - \sqrt{3}\right) = Mv^2$ ——— ②

Divide ② by ①: $v^2 = \dfrac{10Fg(10-\sqrt{3})}{\frac{F}{2}(10\sqrt{3}+1)}$

$= 20g\left(\dfrac{10-\sqrt{3}}{10\sqrt{3}+1}\right)$

So $v \geq \sqrt{20g\left(\dfrac{10-\sqrt{3}}{10\sqrt{3}+1}\right)}$

Thus $9.50 \leq v \leq 11.99$ m/s.
($57 \leq v \leq 72$ km/h.)

b) (i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

and $(\cos\theta + i\sin\theta)^3 = \cos3\theta + i\sin3\theta$.

Equating coefficients gives:

$$\begin{aligned}\cos3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3(\cos\theta(1 - \cos^2\theta))\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

(ii) Since $\cos3\theta = \frac{1}{2}$, $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$$\Rightarrow 8\cos^3\theta - 6\cos\theta - 1 = 0$$

When $\cos\theta = x$, this equation becomes $8x^3 - 6x - 1 = 0$.

(iii) $\cos3\theta = \frac{1}{2}$

$3\theta = 2n\pi \pm \frac{\pi}{3}$, $n = 0, \pm1, \pm2, \ldots$

$\therefore\ \theta = \frac{\pi}{9}(6n \pm 1)$

Hence the solutions are $x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$.

(iv) since $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$ and $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$ we have

$$\cos\frac{\pi}{9}\cdot\cos\frac{5\pi}{9}\cdot\cos\frac{7\pi}{9} = \frac{1}{8}$$

$$\Rightarrow \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}.$$

Question 6

a) (i) $\frac{d}{dx}\log_e(\sec x + \tan x) = \frac{1}{\sec x + \tan x} \times \left[-(\cos x)^{-2}(-\sin x) + \sec^2 x\right]$

$= \frac{1}{\sec x + \tan x} \times \left[\frac{\sin x}{\cos^2} + \sec^2 x\right]$

$= \frac{1}{\sec x + \tan x}(\tan x \sec x + \sec^2 x)$

$= \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x}$

$= \sec x.$

(ii) $\int_0^{\pi/4} \sec x \, dx = \left[\log_e(\sec x + \tan x)\right]_0^{\pi/4}$

$= \log_e(\sqrt{2}+1) - \log_e 1$

$= \log_e(\sqrt{2}+1)$

(iii) $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$

$= \int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x \, dx$

$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} (n-2)\sec^{n-2} x \tan^2 x \, dx$

$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^n x - \sec^{n-2} x \, dx$

$= (\sqrt{2})^{n-2} - (n-2)(I_n - I_{n-2})$

$\therefore I_n(1+(n-2)) = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$

$I_n = \frac{1}{n-1}\left((\sqrt{2})^{n-2} + (n-2)I_{n-2}\right)$

(iv) $I_3 = \frac{1}{2}(\sqrt{2} + I_1)$

$= \frac{1}{2}(\sqrt{2} + \log(\sqrt{2}+1))$, from (ii) above.

b) $\frac{y^2}{a^2} - \frac{x^2}{a^2} = 1$

$y = \sqrt{a^2 + x^2}$

So $Area = \int_{-a}^{a} \sqrt{a^2+x^2}\, dx$.

Let $x = a\tan\theta$,

$dx = a\sec^2\theta\, d\theta$.

When $x = a$, $\tan\theta = 1$

$\therefore \theta = \frac{\pi}{4}$

So $A = 2\int_0^{\frac{\pi}{4}} \sqrt{a^2 + a^2\tan^2\theta}\,.\, a\sec^2\theta\, d\theta$

$= 2a^2 \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$

$= 2a^2 \times \frac{1}{2}\left(\sqrt{2} + \log_e(\sqrt{2}+1)\right)$ from part (a)

$= a^2\left(\sqrt{2} + \log_e(\sqrt{2}+1)\right)$, as required

(c) $x^2 = 16\left(1 - \frac{y^2}{9}\right)$

since $x = a$, $\delta V = x^2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\delta y$

$= 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\left(1 - \frac{y^2}{9}\right)\delta y$

So $V = 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\int_{-3}^{3}\left(1 - \frac{y^2}{9}\right) dy$

$= 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\left[\left(y - \frac{y^3}{27}\right)\right]_{-3}^{3}$

$= 16\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)(2 + 2)$

$= 64\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$

QUESTION 7

a)(i) In ΔPBT, PTC

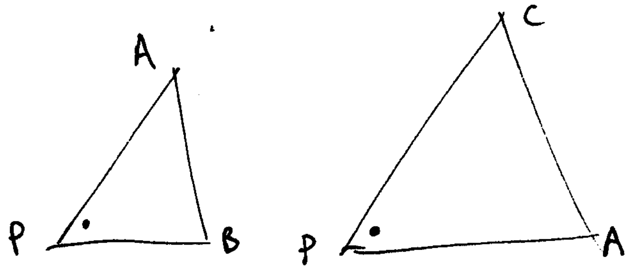
$\angle BTP = \angle PCT$ (angle in alternate segment)

$\angle BPT = \angle TPC$ (common angles)

$\therefore \Delta PBT \parallel\!| \Delta PTC$ (AA)

(ii) In ΔAPB, PTC

$\angle APB = \angle CPA$ (common)



$PB.BC = PT^2$ (secant/tangent)

$\therefore PB.BC = PA^2$ ($PA = PT$)

$\frac{PA}{PB} = \frac{PC}{PA}$

So $\frac{PA}{PC} = \frac{PB}{PA}$

$\therefore \Delta APB \parallel\!| \Delta PTC$ (SAS)

(iii) $\angle PAB = \angle PCA$ (from part (ii) – corresponding angles are equal)

$\therefore \angle PAB = \alpha$.

$\angle BED = 180 - \alpha$ (opposite angles in cyclic quadrilateral are supplementary)

$\therefore \angle AED = \alpha$ (straight angles).

So $\angle PAB = \angle AEB$ and since the angles are alternate and equal, $AP \parallel DE$.

Question 7

(b) (i) $\int_n^{2n} \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_n^{2n}$

$= 2\sqrt{2n} - 2\sqrt{n}$

$= 2\sqrt{n}\left(\sqrt{2}-1\right)$

(ii) The upper rectangles have area greater than the integral so,

$$\int_n^{2n} \frac{dx}{\sqrt{x}} < \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n-1}} .$$

Adding $\frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{n}}$ to both sides and using (i) gives

$$2\sqrt{n}\left(\sqrt{2}-1\right) + \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{2n}} \quad \text{———} \; *$$

The lower rectangles have area less than the integral so,

$$\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \dots + \frac{1}{\sqrt{2n}} < \int_n^{2n} \frac{dx}{\sqrt{x}} \quad \text{———} \; **$$

From * and ** we get

$$2\sqrt{n}\left(\sqrt{2}-1\right) + \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{n}} < S_n < 2\sqrt{n}\left(\sqrt{2}-1\right)$$

ie/ $2\sqrt{n}\left(\sqrt{2}-1\right) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\left(\sqrt{2}-1\right)$

(iii) When $n = 10^8$, $S_n \doteqdot 8284.2712$

QUESTION 8

(a) (i) $\dfrac{(a+b)^2}{4} - ab = \dfrac{a^2+2ab+b^2}{4} - ab$

$= \dfrac{a^2-2ab+b^2}{4}$

$= \dfrac{(a-b)^2}{4}$

$\geqslant 0$, since $(a+b)^2 \geqslant 0$.

(ii) Applying the result for n positive numbers, we

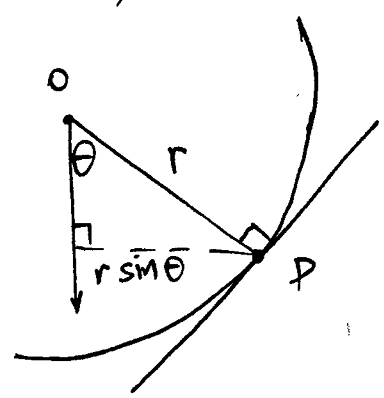
$$\frac{1+2+3+\dots+n}{n} \geqslant \sqrt[n]{1\times 2\times 3\times \dots \times n}$$

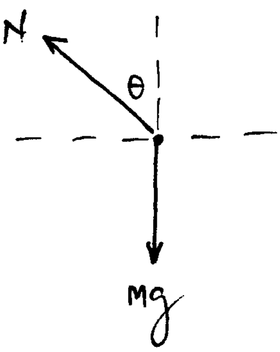
LHS $= \dfrac{n(n+1)}{2}$ and the RHS $= \sqrt[n]{n!}$ and thus,

$$\sqrt[n]{n!} \leqslant \frac{n(n+1)}{2n}$$

ie, $n! \leqslant \left(\dfrac{n+1}{2}\right)^n$.

(b)





Resolving at P:

$N\sin\theta = \dfrac{mv^2}{r\sin\theta}$ —— ①

$N\cos\theta - mg = 0$ —— ②

a) Dividing ① by ② $\tan\theta = \dfrac{\cancel{m}v^2}{r\sin\theta} \times \dfrac{1}{\cancel{m}g}$

$\therefore v^2 = gr\sin\theta\tan\theta$ as required.

b) (see over)

(b) (cont)

From ①, $N \sin^2\theta = \frac{mv^2}{r}$

From ②, $N^2 \cos^2\theta = m^2 g^2$

$$N \cos^2\theta = \frac{m^2 g^2}{N}$$

Adding ① and ② $\quad N = \frac{mv^2}{r} + \frac{m^2 g^2}{N}$

$$N^2 - \frac{mv^2}{r} N - m^2 g^2 = 0$$

$$rN^2 - mv^2 N - rm^2 g^2 = 0$$

Now $\quad \Delta = m^2 v^4 + 4r^2 m^2 g^2$

$$= m^2 (v^4 + 4r^2 g^2)$$

$$\therefore N = \frac{mv^2 + m\sqrt{v^4 + 4r^2 g^2}}{2r}$$ only since $N > 0$.

$$= \frac{m}{2r}\left(v^2 + \sqrt{v^4 + 4r^2 g^2}\right).$$

Question 8

a, c)(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx$

$= \left[-\sin x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx$

$\therefore \quad I_n (1 + (n-1)) = (n-1) I_{n-2}$

$I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$

(ii) $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = I_{2n}$

$= \frac{2n-1}{2n} \times I_{2n-2}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times I_{2n-4}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{1}{2} \times I_0$

$= \frac{2n}{2n} \times \frac{2n-1}{2n} \times \frac{2n-2}{2n-2} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{2}{2} \times \frac{1}{2} \times \int_0^{\frac{\pi}{2}} dx$

$= \frac{(2n)!}{4(n)^2 \times 4(n-1)^2 \times 4(n-2)^2 \times \cdots \times 2^2} \times \frac{\pi}{2}$

$= \frac{(2n)!}{4^n (n!)^2} \times \frac{\pi}{2}$

$= \frac{\pi (2n)!}{2^{2n+1} (n!)^2}.$