SYDNEY GRAMMAR SCHOOL TRIAL EXAMINATION 2001

3/4 UNIT MATHEMATICS FORM VI

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 13th August 2001

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the left margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

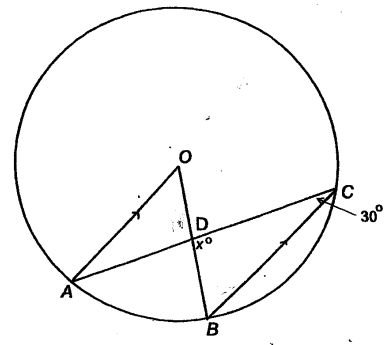
QUESTION ONE (Start a new answer booklet)

Marks

2 (a) Find the coordinates of the point that divides the interval joining the points $(-5, 6)$ and $(4, -3)$ in the ratio $3 : 1$.

3 (b) Find the acute angle between the lines $x + 2y = 5$ and $x - 3y = 0$.

(c)



In the diagram above, O is the centre of the circle, $BC \parallel AO$ and $\angle ACB = 30°$.

1 (i) Explain why $\angle AOB = 60°$.

2 (ii) Find x, giving reasons.

(d) Consider the polynomial $P(x) = x^3 - x^2 - 10x - 8$.

1 (i) Show that $x = -1$ is a zero of $P(x)$.

2 (ii) Express $P(x)$ as a product of three linear factors.

1 (iii) Solve $P(x) \leq 0$.

Exam continues next page ...

QUESTION TWO (Start a new answer booklet)

Marks

[1] (a) Sketch the polynomial function $y = x^2(x^2 - 16)$, carefully showing all intercepts.

[1] (b) (i) Write $x^2 + 4x + 5$ in the form $(x + a)^2 + b$.

[2] (ii) Hence find $\int \frac{dx}{x^2 + 4x + 5}$.

[3] (c) Find the general solution of $\cos 2x = \cos x$.

[2] (d) (i) Sketch the parabola $f(x) = 9 - (x + 2)^2$, showing clearly any intercepts with the axes and the coordinates of the vertex.

[1] (ii) What is the largest domain containing the value $x = 0$ for which the function has an inverse function?

[2] (iii) On a separate diagram, sketch the graph of this inverse function, showing all intercepts with the axes.

QUESTION THREE (Start a new answer booklet)

Marks

[2] (a) Evaluate $\lim_{x \to 0} \left(\frac{\sin 4x}{\tan 2x} \right)$. You must show all working for full marks.

[2] (b) Find the term independent of x in the expression $\left(x + \frac{1}{x^2} \right)^9$.

[4] (c) A spherical balloon is expanding so that its volume $V\,\mathrm{m}^3$ increases at a constant rate of $72\,\mathrm{m}^3$ per second. What is the rate of increase of the surface area when the radius is 12 metres? You may use the formulae $V = \frac{4}{3}\pi r^3$ for the volume of a sphere and $S = 4\pi r^2$ for its surface area.

[1] (d) (i) Show that there is a root to the equation $\sin x = x - \frac{1}{2}$ between $x = 0{\cdot}5$ and $x = 1{\cdot}8$.

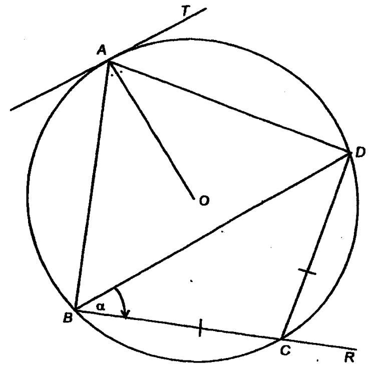
[3] (ii) Taking $x = 1{\cdot}2$ as a first approximation to this solution, apply Newton's method once to find a closer approximation to the solution. Give your answer correct to two decimal places.

QUESTION FOUR (Start a new answer booklet)

Marks

[2] (a) Write $3\sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$, where $0 \le \alpha \le \frac{\pi}{2}$.

(b)



In the diagram above, the points A, B, C and D lie on a circle with centre O. The line TA is a tangent to the circle. The chord BC is produced to R. The interval AO bisects $\angle BAD$ and $BC = CD$.
Let $\angle DBC = \alpha$.

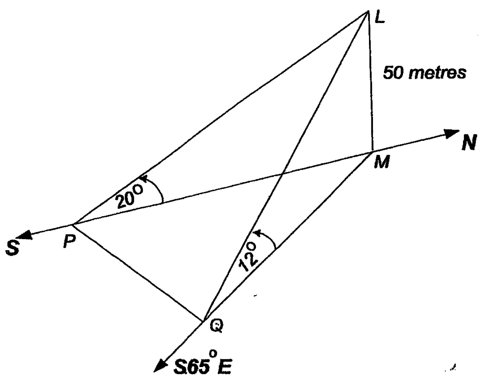
Copy the diagram onto your answer paper.

[2] (i) Prove that $\angle DCR = 2\alpha$.

[1] (ii) Show that $\angle OAD = \alpha$.

[2] (iii) Prove that $\angle ABC$ is a right angle.

(c)



From the top L of a lighthouse 50 metres high a boat is observed at a point P due south at an angle of depression of 20°, as shown in the diagram above. The boat drifts at a constant speed and in a constant direction. After 10 minutes it is again observed from the top of the lighthouse at the point Q at an angle of depression of 12°. The base M of the lighthouse is at sea-level, and the bearing of Q from M is $S65^\circ E$.

$\boxed{1}$ (i) Find an expression for PM.

$\boxed{3}$ (ii) Show that the distance PQ is given by

$$PQ = 50\sqrt{\cot^2 20^\circ + \cot^2 12^\circ - 2\cot 20^\circ \cot 12^\circ \cos 65^\circ}.$$

$\boxed{1}$ (iii) How fast was the boat drifting? Give your answer in metres per second, correct to two significant figures.

QUESTION FIVE (Start a new answer booklet)

Marks

$\boxed{2}$ (a) (i) Differentiate $x\cos^{-1}x - \sqrt{1-x^2}$.

$\boxed{1}$ (ii) Hence evaluate $\displaystyle\int_0^1 \cos^{-1}x\,dx$.

$\boxed{5}$ (b) Use the substitution $u = 1 - x$ to evaluate $\displaystyle\int_{-3}^0 \frac{x}{\sqrt{1-x}}\,dx$.

$\boxed{4}$ (c) By considering the expansion of $(1+x)^{2n} = (1+x)^n(1+x)^n$ in two different ways, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

THE EXAMINATION PAPER CONTINUES ON THE NEXT PAGE

QUESTION SIX (Start a new answer booklet)

(a) Let $(3+2x)^{20} = \sum_{r=0}^{20} a_r x^r$.

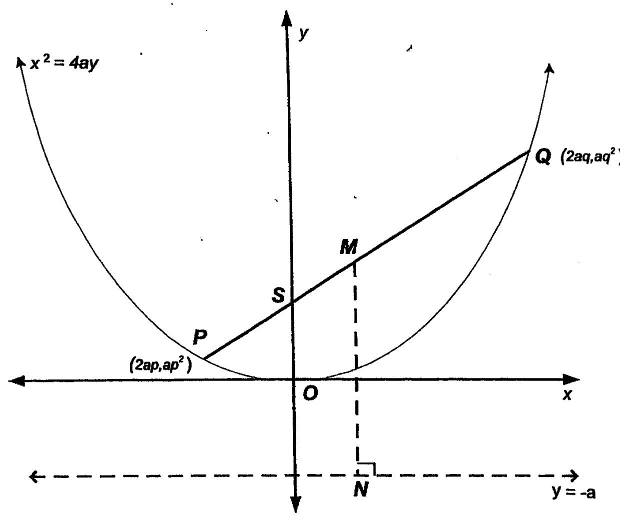
Marks

$\boxed{1}$ (i) Write an expression for a_r.

$\boxed{1}$ (ii) Show that $\dfrac{a_{r+1}}{a_r} = \dfrac{40-2r}{3r+3}$.

$\boxed{4}$ (iii) Hence find the greatest coefficient in the expansion of $(3+2x)^{20}$.

(b)



Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be points on the parabola $x^2 = 4ay$, as shown in the above diagram.

$\boxed{1}$ (i) Show that the equation of the chord PQ is $y = \dfrac{p+q}{2}x - apq$.

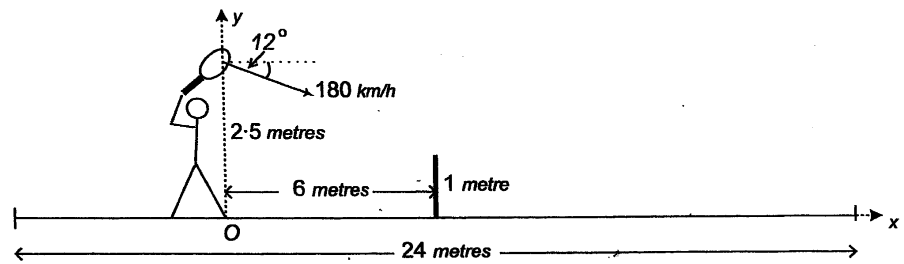
$\boxed{1}$ (ii) Show that if the chord PQ passes through the focus $S(0, a)$, then $pq = -1$.

$\boxed{4}$ (iii) M is the midpoint of the focal chord PQ. N lies on the directrix such that MN is perpendicular to the directrix. T is the midpoint of MN. Find the locus of T.

Exam continues overleaf ...

QUESTION SEVEN (Start a new answer booklet)

(a)



In the diagram above, a tennis court is 24 metres long and has a net one metre high positioned in the middle.
During a match a player standing 6 metres from the net smashes a ball into the opposing court with an initial speed of 180 km/h. The ball is hit parallel to the sideline and is projected with an angle of depression of 12° from a height of 2·5 metres above the ground. Let $g = 10\,\text{m/s}^2$.

Marks

$\boxed{3}$ (i) Taking the axes as given on the diagram, show that the horizontal and vertical components of the displacement are given by

$$x = 50t\cos 12^\circ \qquad \text{and} \qquad y = -5t^2 - 50t\sin 12^\circ + 2{\cdot}5$$

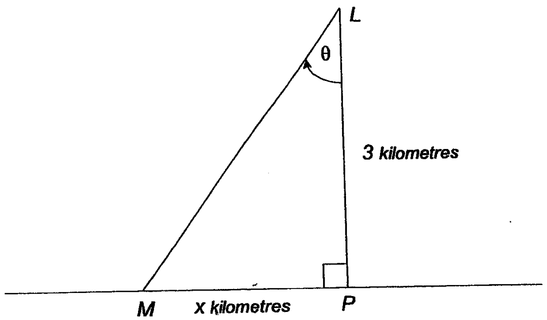
respectively, where t is the time in seconds and both x and y are measured in metres.

$\boxed{2}$ (ii) By what margin does the ball clear the net? Give your answer correct to the nearest centimetre.

$\boxed{2}$ (iii) How far from the opposing court's baseline does the ball land? Give your answer correct to the nearest centimetre.

Exam continues next page ...

(b)



In the diagram above, a lighthouse L containing a revolving beacon is located out at sea, 3 kilometres from P, the nearest point on a straight shoreline. The beacon rotates clockwise with a constant rotation rate of 4 revolutions per minute and throws a spot of light onto the shoreline.
When the spot of light is at M, x km from P, the angle at L is θ.

1 (i) Explain why $\dfrac{d\theta}{dt} = 8\pi$, where t is the time measured in minutes.

2 (ii) How fast is the spot moving when it is at P?

2 (iii) How fast is the spot moving when it is at a point on the shoreline 2 km from P?

JCM

2001 3 UNIT TRIAL MARKING SCHEME

QUESTION 1

(a) $x = \dfrac{3\times 4 + 1\times(-5)}{3+1}$

$= \dfrac{7}{4}$ ✓

$y = \dfrac{3\times(-3) + 1\times 6}{4}$

$= -\dfrac{3}{4}$ ✓

the point is $\left(\dfrac{7}{4}, -\dfrac{3}{4}\right)$

(b) $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{3}$

let θ be the acute angle

$\tan\theta = \left|\dfrac{-\frac{1}{2} - \frac{1}{3}}{1 + (-\frac{1}{2})(+\frac{1}{3})}\right|$ ✓✓

$\theta = 45^\circ$ ✓

(c) (i) the angle at the centre is equal to twice the angle at the circumference when they are subtended by the same arc. ✓

(ii) $\angle OBC = 60^\circ$ (alternate angles, $AO \parallel BC$) ✓

$x = 90$ (angle sum of ΔBCD) ✓

(d) $P(x) = x^3 - x^2 - 10x - 8$

(i) $P(-1) = -1 - 1 + 10 - 8 = 0$

so $x = -1$ is a zero of $P(x)$ ✓

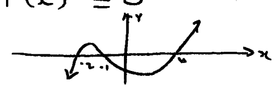
(ii) $(x+1)$ is a factor of $P(x)$

$$\begin{array}{r} x^2 - 2x - 8 \\ x+1 \overline{)\, x^3 - x^2 - 10x - 8} \\ x^3 + x^2 \\ \hline -2x^2 - 10x \\ -2x^2 - 2x \\ \hline -8x - 8 \\ -8x - 8 \\ \hline 0 \end{array}$$ ✓

$P(x) = (x+1)(x^2 - 2x - 8)$

$= (x+1)(x-4)(x+2)$ ✓

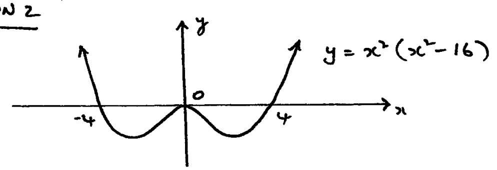
(iii) $P(x) \le 0$



$x \le -2$ or $-1 \le x \le 4$ ✓

QUESTION 2

(a)



✓

(b) (i) $x^2+4x+5 = (x+2)^2+1$ ✓

(ii) $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{1+(x+2)^2}$

$= \tan^{-1}(x+2) + C$ ✓✓

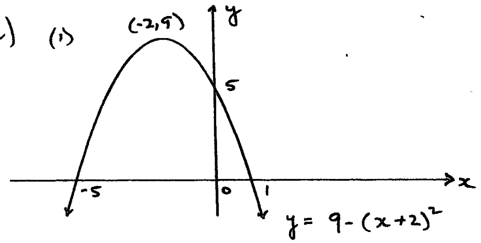
(c) $\cos 2x = \cos x$

$2x = 2n\pi \pm x$ ✓

$3x = 2n\pi$ or $x = 2n\pi$ ✓

$x = \dfrac{2n\pi}{3}$ for any integer n ✓

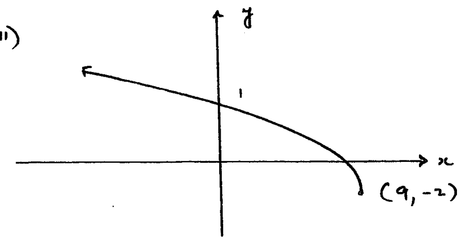
(d) (i)



✓✓

(ii) $x \geq -2$ ✓

(iii)



✓✓

QUESTION 3

(a) $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{\tan 2x} = \lim_{x\to 0} \frac{\sin 4x}{4x} \times \lim_{x\to 0} \frac{2x \times 2}{\tan 2x}$ ✓

$= 2$ ✓

(b) $\left(x + \frac{1}{x^2}\right)^9$

$T_r = {}^9C_r\, x^r \,(x^{-2})^{9-r}$

$= {}^9C_r\, x^{3r-18}$ ✓

for the term independent of x

$3r - 18 = 0$

$r = 6$

Hence the term is ${}^9C_6 = 84$ ✓

(c) $\dfrac{dv}{dt} = 72$

$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

$\dfrac{dv}{dr} = 4\pi r^2$, $\dfrac{ds}{dr} = 8\pi$

$\dfrac{ds}{dt} = \dfrac{ds}{dr} \times \dfrac{dv}{dt} \times \dfrac{dr}{dv}$ ✓

$= \dfrac{8\pi r \times 72}{4\pi r^2}$ ✓✓

$= \dfrac{2 \times 72}{r}$

when $r = 12$ $\dfrac{ds}{dt} = 12\ m^2/s$ ✓

(d)(i) Consider $f(x) = \sin x - x + \frac{1}{2}$

$f(\cdot 5) > 0$ ✓

$f(1\cdot 8) < 0$

so there is a root between $x = 0\cdot 5$ and $x = 1\cdot 8$

(ii) $f'(x) = \cos x - 1$ ✓

$x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ ✓

$= 1\cdot 2 - \dfrac{f(1\cdot 2)}{f'(1\cdot 2)}$

$= 1\cdot 56$ (2 decimal places) ✓

QUESTION 4

(a) $3\sin x + \sqrt{3}\cos x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\sin\alpha = \sqrt{3}$

$R\cos\alpha = 3$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \pi/6$ ✓

$R = \sqrt{3^2+\sqrt{3}^2}$

$= 2\sqrt{3}$ ✓

$3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)$

(b) (i) $\angle BDC = \alpha$ (base angles of isosceles Δ) ✓

$\angle DCR = 2\alpha$ (exterior angle of ΔBCD) ✓

(ii) $\angle BAD = 2\alpha$ (exterior angle of cyclic quad. ABCD) ✓

$\therefore \angle OAD = \alpha$ (OA bisects $\angle BAD$)

(iii) $OA \perp AT$ (radius is perpendicular to the tangent at the point of contact) ✓

so, $\angle TAD = 90^\circ - \alpha$

$\angle ABD = \angle TAD$ (alternate segment theorem) ✓

so, $\angle ABC = (90^\circ - \alpha) + \alpha$

$= 90^\circ$

(c)(i) In ΔLMP: $\tan 20^\circ = \frac{LM}{PM}$

$PM = 50\cot 20^\circ$ metres ✓

(ii) $PQ^2 = PM^2 + QM^2 - 2.PM.QM.\cos\angle PMQ$ (cosine rule)

$= 50^2\cot^2 20^\circ + 50^2\cot^2 12^\circ - 2.50\cot 20^\circ.\cot 12^\circ.\cos 65^\circ$ ✓✓

so, $PQ = 50\sqrt{\cot^2 20^\circ + \cot^2 12^\circ - 2\cot 20^\circ\cot 12^\circ.\cos 65^\circ}$ ✓

(iii) Speed $= \frac{PQ}{10\times 60}$

$= 0.36$ m/s (2 sig. fig.) ✓

QUESTION 5

(a) (i) $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$

$= \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} - \frac{-\frac{1}{2}2x}{\sqrt{1-x^2}}$ ✓✓

$= \cos^{-1}x$

(ii) $\int \cos^{-1}x\,dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= 1$ ✓

(b) $u = 1-x \Rightarrow x = 1-u$

$du = -dx$

when $x=-3$ $u=4$

when $x=0$ $u=1$ ✓

$I = \int_4^1 \frac{1-u}{\sqrt{u}}\,-du$ ✓✓

$= \int_1^4 u^{-1/2} - u^{1/2}\,du$ ✓

$= \left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]_1^4$

$= \left(4 - \frac{2}{3}\times 4\times 2\right) - \left(2 - \frac{2}{3}\right)$

$= -8/3$ ✓

(c) for $(1+x)^{2n}$

the coefficient of x^n is $\binom{2n}{n}$ ✓

$(1+x)^{2n} = (1+x)^n(1+x)^n$

$= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right]\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}\right]$

the coefficient of x^n is: $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$ ✓

since $\binom{n}{r} = \binom{n}{n-r}$ then the coefficient of x^n is

$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$ ✓

Equating the co-efficients of x^n gives

$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^n = \binom{2n}{n}$

QUESTION 6

(a) (i) $(3+2x)^{20} = \sum_{r=0}^{20} {}^{20}C_r \, 3^{20-r} (2x)^r$

so, $a_r = {}^{20}C_r \, 3^{20-r} 2^r$ ✓

(ii) $\frac{a_{r+1}}{a_r} = \frac{{}^{20}C_{r+1} \, 3^{19-r} 2^{r+1}}{{}^{20}C_r \, 3^{20-r} 2^r}$

$= \frac{20-r}{r+1} \times \frac{2}{3}$ ✓

$= \frac{40-2r}{3r+3}$

(iii) let $\frac{a_{r+1}}{a_r} > 1$ ✓

then, $\frac{40-2r}{3r+3} > 1$

$40-2r > 3r+3$

$5r < \cancel{\#}37$

$r < 7\frac{2}{3}$ ✓

when $r=7$: $a_8 > a_7$

$r=6$: $a_7 > a_6$

$\vdots$

$r=0$: $a_1 > a_0$

i.e. $a_8 > a_7 > a_6 > \cdots > a_0$

if $\frac{a_{r+1}}{a_r} < 1$ then $a_8 > a_9 > \cdots > a_{20}$ ✓

So the greatest co-efficient is $a_8 = {}^{20}C_8 \, 3^{12} 2^8$ ✓

(b) (i) $M_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{p+q}{2}$

equation of PQ: $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

so, $y = \frac{p+q}{2}x - apq$ ✓

(ii) If $S \in PQ$ then when $x=0$, $y=a$

i.e. $a = 0 - apq$

so, $pq = -1$ ✓

(iii) M is $\left(a(p+q), \frac{ap^2+aq^2}{2}\right)$

N is $(a(p+q), -a)$ ✓

so T is $\left(a(p+q), \frac{ap^2+aq^2-2a}{4}\right)$

The locus of T is

$x = a(p+q)$ — (1)

$y = \frac{a}{4}(p^2+q^2-2)$ —(2) ✓

from (ii) $pq = -1$, $y = \frac{a}{4}(p^2+q^2+2pq)$ ✓

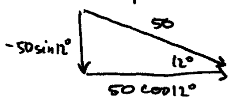
i.e. $y = \frac{a}{4}(p+q)^2$

so, $y = \frac{a}{4}\frac{x^2}{a^2}$ from (1)

i.e. $x^2 = 4ay$ ✓

QUESTION 7

(a)(i) $180 \text{ km/h} = 50 \text{ m/s}$



$\dot{x} = 50\cos 12°$

$x = 50t\cos 12° + c_1$

when $t=0$, $x=0$

so, $x = 50t\cos 12°$

$\ddot{y} = -10$

$\dot{y} = -10t + c_2$

when $t=0$ $\dot{y} = -50\sin 12°$

so, $\dot{y} = -10t - 50\sin 12°$

$y = -5t^2 - 50t\sin 12° + c_3$

when $t=0$ $y = 2.5$

so, $y = -5t^2 - 50t\sin 12° + 2.5$

(ii) when $x=6$ $t = \frac{6}{50\cos 12°}$

when $t = \frac{6}{50\cos 12°}$ $y = 1.149\ldots$

so the ball clears the net by 15cm.

(iii) when $y=0$, $5t^2 + 50t\sin 12° - 2.5 = 0$

$t = \frac{-50\sin 12° \pm \sqrt{(50\sin 12°)^2 + 50}}{10}$

when $t = \frac{-50\sin 12° + \sqrt{(50\sin 12°)^2 + 50}}{10}$

$x = 10.6468\ldots$

So it lands 7.35 metres from the base line

(b) (i) 4 revs/min $= 8\pi$ rad/min

so, $\frac{d\theta}{dt} = 8\pi$

(ii) $\tan\theta = \frac{x}{3}$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$

$= 3\sec^2\theta \cdot 8\pi$

$= 24\pi\sec^2\theta$.

at P $\theta = 0$

so $\frac{dx}{dt} = 24\pi$ km/min.

(iii) when $x=2$, $\cos\theta = \frac{3}{\sqrt{13}}$

so, $\frac{dx}{dt} = \frac{24\pi}{\left(\frac{3}{\sqrt{13}}\right)^2}$

$= \frac{104\pi}{3}$ km/min.