SOLUTIONS - 3UNIT TRIAL - 1996 - JRAHS (1)

Question 1

a) (i) $\frac{d}{dx}\left(x^2\ln(1+x^2)\right)$

[2] $= x^2\cdot\frac{2x}{1+x^2} + \ln(1+x^2)\cdot 2x$

$= \frac{2x^3}{1+x^2} + 2x\ln(1+x^2)$

(ii) $\frac{d}{dx}\left(\tan^{-1}x\right)^2$

$= 2(\tan^{-1}x)\cdot\frac{1}{1+x^2}$ [2]

$= \frac{2\tan^{-1}x}{1+x^2}$

b) (i) $\int_1^2 \frac{x^2+1}{x}\,dx$

$= \int_1^2 \left(x+\frac{1}{x}\right)dx$ [3]

$= \left[\frac{x^2}{2}+\ln x\right]_1^2$

$= 2+\ln 2 - \frac{1}{2} - \cancel{\ln 1}$

$= \ln 2 + \frac{3}{2}$

(ii) $\int_1^{\sqrt3}\frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt3}$

$= \sin^{-1}\frac{\sqrt3}{2} - \sin^{-1}1$

$= \frac{\pi}{3} - \frac{\pi}{2}$ [2]

$= -\frac{\pi}{6}$

c) (i) $\sin(x-y) = \sin x\cos y - \cos x\sin y$ [1]

(ii) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{\sqrt3}{2}\cdot\frac{1}{\sqrt2} - \frac{1}{2}\cdot\frac{1}{\sqrt2}$ [2]

$= \frac{\sqrt3}{2}\cdot\frac{\sqrt2}{2} - \frac{1}{2}\cdot\frac{\sqrt2}{2}$

$= \frac{\sqrt6-\sqrt2}{4}$

Question 2

a) $\int_0^{1/2} 2x(1-2x)^4\,dx$

$= \int_1^0 2\left(\frac{1-u}{2}\right)u^4\cdot -\frac{1}{2}\,du$

$= \int_0^1 \frac{1}{2}(1-u)u^4\,du$ [1]

$= \frac{1}{2}\left[\frac{u^5}{5}-\frac{u^6}{6}\right]_0^1$ [1]

$= \frac{1}{2}\left(\frac{1}{5}-\frac{1}{6}\right)$

$= \frac{1}{60}$ [1]

$u = 1-2x$

$x = \frac{1-u}{2}$

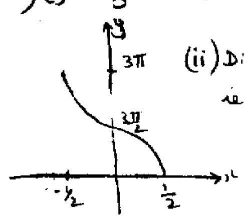
$dx = -\frac{1}{2}du$

$x=0 \Rightarrow u=1$ [1]

$x=\frac{1}{2} \Rightarrow u=0$

total [5]

b) (i) $\frac{y}{3} = \cos^{-1}2x$



(ii) D: $-1 \le 2x \le 1$

ie $-\frac{1}{2} \le x \le \frac{1}{2}$

R: $0 \le \frac{y}{3} \le \pi$

ie. $0 \le y \le 3\pi$ [4]

(2)

2 c) $V = \pi\int_{-2}^{2}\frac{1}{4+x^2}\,dx$

$= \frac{\pi}{2}\left[\tan^{-1}\frac{x}{2}\right]_{-2}^{2}$

$= \frac{\pi}{2}\left[\tan^{-1}1 - \tan^{-1}(-1)\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$

$= \frac{\pi^2}{4}$ [1]

b) (i) $P(4G) = \frac{^8C_4}{^{24}C_4} = \frac{70}{10624} = \frac{5}{759}$

OR

[2] $P(GGGG) = \frac{8}{24}\cdot\frac{7}{23}\cdot\frac{6}{22}\cdot\frac{5}{21}$

$= \frac{5}{759}$

(ii) P(at least 2 Green)

$= P(2G,2O) + P(3G,1O) + P(4G)$

$= \frac{^8C_2\cdot{}^{16}C_2}{^{24}C_4} + \frac{^8C_3\cdot{}^{16}C_1}{^{24}C_4} + \frac{^8C_4}{^{24}C_4}$

$\doteq 0.407$

[2]

Question 3

a) (i) $\cos x - \sqrt3\sin x = A\cos(x+\alpha)$

$= A\cos x\cos\alpha - A\sin x\sin\alpha$

$\therefore 1 = A\cos\alpha$ $\quad \sqrt3 = A\sin\alpha$

$\frac{1}{A} = \cos\alpha$ $\quad \frac{\sqrt3}{A} = \sin\alpha$

$\therefore \frac{1}{A^2} + \frac{3}{A^2} = 1$

$4 = A^2$

$2 = A$

$\tan\alpha = \sqrt3$ [2]

$\alpha = \frac{\pi}{3}$

$\therefore \cos\alpha - \sqrt3\sin\alpha = 2\cos\left(x+\frac{\pi}{3}\right)$

(ii) Solve $\cos\alpha - \sqrt3\sin\alpha = 1$

ie $2\cos\left(x+\frac{\pi}{3}\right) = 1$

$\cos\left(x+\frac{\pi}{3}\right) = \frac{1}{2}$

$x+\frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$, $n\in J$

ie. $x = 2n\pi - \frac{2\pi}{3}$ or $2n\pi$, $n \in J$

Alternatively

P(at least 2 Green)

$= 1 - \left[P(0\text{ green}) + P(1\text{ green})\right]$

$= 1 - \left[\frac{^{16}C_4}{^{24}C_4} - \frac{^8C_1\cdot{}^{16}C_3}{^{24}C_4}\right]$

$= 0.407$

c) $T_{k+1} = {}^{30}C_k\cdot(2x)^{30-k}\cdot\left(-\frac{1}{x^2}\right)^k$

$= {}^{30}C_k\cdot 2^{30-k}\cdot x^{30-k}\cdot(-x)^{-2k}$

$= {}^{30}C_k\cdot 2^{30-k}\cdot(-1)^k\cdot x^{30-k-2k}$

ie $30-k-2k = 12$

$-3k = -18$ [3]

$k = 6$

$\therefore$ Coefficient of x^{12} term is:

${}^{30}C_6\cdot 2^{24}\cdot(-1)^6$

$= 30.\ .\ 2^{24}$

Question 4

c) (i) $N = 5000 + Ae^{kt}$ [2]

$\frac{dN}{dt} = k \cdot Ae^{kt}$

$= k(N - 5000)$

(ii) When $t = 0$, $N = 15000$:

$N = 5000 + Ae^{kt}$

$15000 = 5000 + Ae^{0}$

$\underline{10000 = A}$

$\therefore N = 5000 + 10000e^{kt}$

When $t = 2$, $N = 20000$:

$20000 = 5000 + 10000e^{k \times 2}$

$\frac{15000}{10000} = e^{2k}$ [3]

$\ln 1.5 = 2k$

$\frac{\ln 1.5}{2} = k$

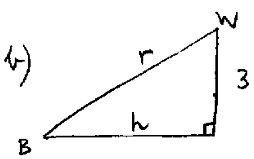
$(k \doteqdot 0.20273)$

(iii) $N = 5000 + 10000e^{7 \times \frac{\ln 1.5}{2}}$

$N = \underline{46335}$ (nearest number)

[2]

b)



$\frac{dr}{dt} = 12$ m/s

$\frac{dh}{dt} = ?$ when $h = 5$

$h^2 + 9 = r^2$

When $h = 5$:

$25 + 9 = r^2$

$\underline{\sqrt{34} = r}$

[5]

$h^2 + 9 = r^2$

$h = \sqrt{r^2 - 9}$

$\frac{dh}{dr} = \frac{2r}{2\sqrt{r^2-9}}$

ie $\frac{dh}{dr} = \frac{r}{\sqrt{r^2-9}}$

When $r = \sqrt{34}$:

$\frac{dh}{dr} = \frac{\sqrt{34}}{5}$

$\therefore$ By Chain Rule:

$\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$

$= \frac{\sqrt{34}}{5} \cdot 12$

$= \frac{12\sqrt{34}}{5}$ m/s

$(\doteqdot 13.99$ m/s$)$

Question 5

a) (i) $\cot x + \tan x$

$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$

$= \frac{1}{\sin x \cos x}$

$= \frac{2}{\sin 2x}$ [2]

$= \underline{2 \operatorname{cosec} 2x}$

(ii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} 2x \, dx$

$= \frac{1}{2}\int_{\pi/6}^{\pi/3} (\cot x + \tan x)\, dx$

$= \frac{1}{2}\int_{\pi/6}^{\pi/3} \left(\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}\right) dx$

$= \frac{1}{2}\left[\ln(\sin x) - \ln(\cos x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{2}\left[\ln\left(\frac{\sin x}{\cos x}\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ [2]

$= \frac{1}{2}\left[\ln(\tan x)\right]_{\pi/6}^{\pi/3}$

$= \frac{1}{2}\left[\ln\left(\tan\frac{\pi}{3}\right) - \ln\left(\tan\frac{\pi}{6}\right)\right]$

$= \frac{1}{2}\left[\ln\sqrt{3} - \ln\left(\frac{1}{\sqrt{3}}\right)\right]$

$= \frac{1}{2}\ln\frac{\sqrt{3}}{\frac{1}{\sqrt{3}}}$

$= \underline{\frac{1}{2}\ln 3}$

(iii) $\cot x + \tan x = 2\operatorname{cosec} 2x$

$\therefore \cot 15 + \tan 15 = 2\operatorname{cosec} 30^\circ$

$\cot 15 + \frac{1}{\cot 15} = 2 \times 2$

$\cot^2 15 - 4\cot 15 + 1 = 0$

Let $y = \cot 15$:

$y^2 - 4y + 1 = 0$

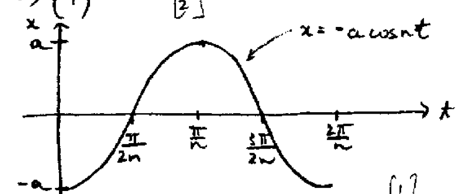
$y = \frac{4 \pm \sqrt{16 - 4.1.1}}{2}$

$y = \frac{4 \pm \sqrt{12}}{2}$ [2]

$y = 2 \pm \sqrt{3}$

$\therefore \underline{\cot 15 = 2 + \sqrt{3}}$ only

b) (i) [2]



Period $= \frac{2\pi}{n}$ [1]

(ii) $x = -a\cos nt$ [1]

$\dot{x} = +an\sin nt$

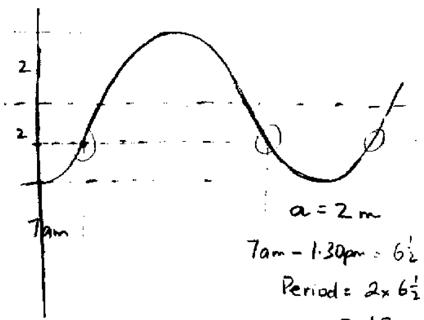
$\ddot{x} = +an^2\cos nt$

$= -n^2(-a\cos nt)$

ie $\ddot{x} = -n^2 x$

$\therefore$ SHM

(iii)



$a = 2\,m$

$7am - 1{:}30pm = 6\frac{1}{2}$

Period $= 2 \times 6\frac{1}{2}$ hr

$= 13$

ie $\frac{2\pi}{n} = 13$

$\frac{2\pi}{13} = n$

$x = -a\cos nt$

ie $x = -2\cos nt$

When $x = -1$ (ie depth $= 8 + 1 = 9m$)

$-1 = -2\cos\frac{2\pi}{13}t$

$\frac{1}{2} = \cos\frac{2\pi}{13}t$

$\frac{2\pi}{13}t = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$t = \frac{13}{2\pi}\times\frac{\pi}{3}, \frac{13}{2\pi}\times\frac{5\pi}{3}, \frac{13}{2\pi}\times\frac{7\pi}{3}$

$t = 2\frac{1}{6}, \quad 10\frac{5}{6}, \quad 15\frac{1}{6}$

Alternatively:

$x = b - a\cos nt$

$9 = 10 - 2\cos\left(\frac{2\pi}{13}t\right)$

$-1 = -2\cos\left(\frac{2\pi}{13}t\right)$

$\therefore$ time boat can proceed is [4]

$7 + 2\frac{1}{6}$ hr $= 9{.}10$ am

$7 + 10\frac{5}{6}$ hr $= 5{.}50$ pm

$\therefore$ 9.10 am — 5.50 pm

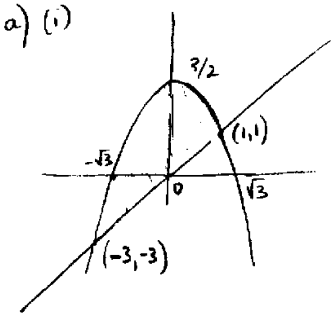
also boat can proceed again at:

$7 + 15\frac{1}{6} = 10{.}10$ pm

$\therefore$ 10.10 — 12 midnight (same day)

Question 6.

a) (i)



2

(ii) $\int_{-3}^{1}\left(\left(\frac{3-x^2}{2}\right) - x\right)\,dx$

$= \frac{1}{2}\int_{-3}^{1}(3 - x^2 - 2x)\,dx$ 2]

$= \left[3x - \frac{x^3}{3} - x^2\right]_{-3}^{1}$

$= \left(3 - \frac{1}{3} - 1\right) - (-9 + 9 - 9)$

$= 1\frac{2}{3} + 9$

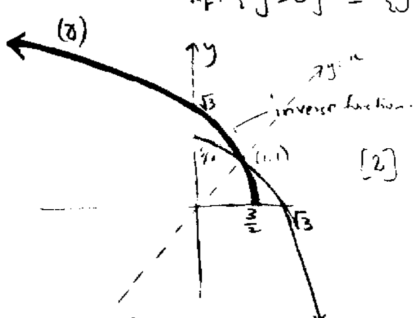
$= 10\frac{2}{3}\,u^2$

(iii) α) $x \geq 0$ Alt Sol^n or $\{x \leq 0\}$

β) $D_F: \{x \leq \frac{3}{2}\}$ or $\{x \leq \frac{3}{2}\}$

$R_F: \{y \geq 0\}$ or $\{y \leq 0\}$

(γ)



[2]

$x = \frac{3-y^2}{2}$

$\Rightarrow 2x = 3 - y^2$

$y^2 = 3 - 2x$

b(i) Using $(1+x)^8(1+x)^8 = (1+x)^{16}$

ie $\left({}^8C_0 + {}^8C_1x + {}^8C_2x^2 + \dots + {}^8C_8x^8\right)\left({}^8C_0 + {}^8C_1x + {}^8C_2x^2 + \dots + {}^8C_8x^8\right) =$

${}^{16}C_0 + {}^{16}C_1x + {}^{16}C_2x^2 + \dots + {}^{16}C_{16}x^{16}$

RHS: Coefficient of x^8 term $= {}^{16}C_8$

LHS: Coefficient of x^8 terms $= {}^8C_0\cdot{}^8C_8 + {}^8C_1\cdot{}^8C_7 + {}^8C_2\cdot{}^8C_6 + \dots$

$\dots + {}^8C_6\cdot{}^8C_2 + {}^8C_7\cdot{}^8C_1 + {}^8C_8\cdot{}^8C_0$

[3] $= ({}^8C_0)^2 + ({}^8C_1)^2 + ({}^8C_2)^2 + \dots + ({}^8C_7)^2 + ({}^8C_8)^2$

as required

(ii) Using the binomial expansion $(q+p)^8$ where $q = p = \frac{1}{2}$ the probability that each of A, B (independently) tosses exactly r heads in 8 tosses of a coin is ${}^8C_r\, q^{8-r}p^r = {}^8C_r\left(\frac{1}{2}\right)^{8-r}\left(\frac{1}{2}\right)^r = {}^8C_r\left(\frac{1}{2}\right)^8$

Since the tossing of r heads by Albert & Bruce are independent events, then the probability that both toss exactly r heads in 8 tosses is:

${}^8C_r\left(\frac{1}{2}\right)^8 \times {}^8C_r\left(\frac{1}{2}\right)^8 = \left({}^8C_r\right)^2\cdot\left(\frac{1}{2}\right)^{16}$

Hence, the probability that Albert and Bruce both toss exactly:

0 heads: $({}^8C_0)^2\left(\frac{1}{2}\right)^{16}$

1 heads: $({}^8C_1)^2\left(\frac{1}{2}\right)^{16}$

2 heads: $({}^8C_2)^2\left(\frac{1}{2}\right)^{16}$

⋮

8 heads: $({}^8C_8)^2\left(\frac{1}{2}\right)^{16}$

Thus, the probability that they toss the same number of heads (ie both 0 heads or 1 head or 2 heads or ... or 8 heads)

$= ({}^8C_0)^2\left(\frac{1}{2}\right)^{16} + ({}^8C_1)^2\left(\frac{1}{2}\right)^{16} + ({}^8C_2)^2\left(\frac{1}{2}\right)^{16} + \dots + ({}^8C_8)^2\left(\frac{1}{2}\right)^{16}$

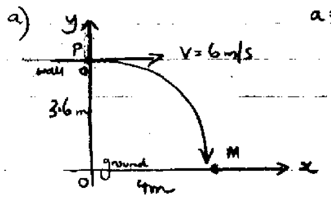
$= \left(\frac{1}{2}\right)^{16}\left[({}^8C_0)^2 + ({}^8C_1)^2 + ({}^8C_2)^2 + \dots + ({}^8C_8)^2\right]$

$= \left(\frac{1}{2}\right)^{16}\cdot{}^{16}C_8$ (from i)

[2]

Question 7

a) $a = 20\,m/s^2 \quad g = 20$



When $t=0$: $x=0, y=0, \dot{x}=6, \ddot{x}=0, y=0, \dot{y}=-20$

(i) $\ddot{x} = 0 \qquad \ddot{y} = -20$

$\dot{x} = 6 \qquad \dot{y} = -20t$

$x = 6t \qquad y = -10t^2 + 3.6$ [2]

(ii) Find t if $y = 0$.

$0 = -10t^2 + 3.6$ [2]

$10t^2 = 3.6$

$t^2 = 0.36$

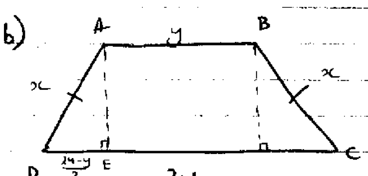
$t = 0.6$ $\quad\therefore$ time taken is 0.6 seconds

(iii) when $t = 0.6$, $x = 6 \times 0.6 = 3.6\,m$

$\therefore\ 4 - 3.6 = 0.4$ [2]

$\therefore$ Kat misses the mouse by 0.4 m

b)



$2x + y = 42$

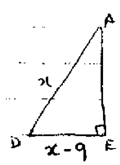
$y = 42 - 2x \quad$ —(1)

[6]

Now: $DE = \dfrac{24-y}{2}$

$= \dfrac{24 - (42-2x)}{2}$ (from (1))

$\therefore\ DE = x - 9$



also: $AE = \sqrt{x^2 - (x-9)^2}$

$= \sqrt{x^2 - x^2 + 18x - 81}$

$\therefore\ AE = \sqrt{18x - 81}$

Let Area of Trapezium be A

$\therefore\ A = \frac{1}{2}(y + 24) \times AE$

$= \frac{1}{2}(42 - 2x + 24)\sqrt{18x - 81}$

$= \frac{1}{2}(66 - 2x)\sqrt{18x - 81}$

$= (33 - x)\sqrt{18x - 81}$

For Max Area, $\dfrac{dA}{dx} = 0$:

$$\frac{dA}{dx} = (33 - x).\tfrac{1}{2}(18x - 81)^{-\frac{1}{2}}.18 + (18x - 81)^{\frac{1}{2}}.(-1)$$

$$= (18x - 81)^{-\frac{1}{2}}\left[9(33 - x) - (18x - 81)\right]$$

$$= (18x - 81)^{-\frac{1}{2}}\left[378 - 27x\right]$$

$$= \frac{378 - 27x}{\sqrt{18x - 81}}$$

ie $\dfrac{dA}{dx} = 0$ when $\dfrac{378 - 27x}{\sqrt{18x - 81}} = 0$

$378 - 27x = 0$

$x = 14$

Check $x = 14$ is a Max:

x	12	14	16
$\frac{dA}{dx}$	4.6	0	-3.7
	/	—	\

$\therefore$ relative max at $x = 14$

as this is a continuous function, then $x = 14$ is the absolute maximum.

Dimensions of Trapezium with greatest area is: