



INTERNATIONAL GRAMMAR SCHOOL
Concordia per Diversitatem

Name:.....

INTERNATIONAL GRAMMAR SCHOOL

MATHEMATICS

Extension 2

YEAR 12

TRIAL EXAMINATION

31st JULY, 2001

Time allowed ---3 hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL eight questions.
- ALL questions are of equal value. (8 @ 15 marks = 120 marks)
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new page*. Number each question clearly.
- Label each page with your name.
- A table of Standard Integrals is attached.

QUESTION 2 (Start a new page)

- a) The complex number Z moves such that $\operatorname{Im} \left(\frac{1}{Z-i} \right) = 1$.
Show that the locus of Z is a circle and find its centre and radius.

3

MARKS

- b) i) Find the square root of the complex number $5 - 12i$
ii) Given that $Z = \frac{1 + \sqrt{5 - 12i}}{2 + 2i}$ and is purely imaginary, find Z^{400}

2

2

- c) i) Shade the region in the argand diagram containing all points representing the complex numbers Z such that $|Z - 1 - i| \leq 1$ and $-\frac{\pi}{4} \leq \operatorname{Arg}(Z - i) \leq \frac{\pi}{4}$
ii) Let ϕ be the complex number of minimum modulus satisfying the inequalities of i). Express ϕ in the form $x + yi$

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- d) Express $\phi = \frac{-1 + i}{\sqrt{3} + i}$ in modulus / argument form.
Hence, evaluate $\cos \frac{7\pi}{12}$ in surd form.

4

QUESTION 1 (Start a new page)

- a) Find $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} dx$

2

- b) i) Find a , b and c such that

$$\frac{16}{(x^2 + 4)(2 - x)} = \frac{ax + b}{x^2 + 4} + \frac{c}{2 - x}$$

2

- ii) Find $\int \frac{16}{(x^2 + 4)(2 - x)} dx$

2

- c) Find $\int \frac{\ln x}{x^2} dx$

4

- d) Use the substitution $t = \tan \frac{\theta}{2}$ to show that

5

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4 \sin \theta - 2 \cos \theta + 6} = \frac{1}{2} \tan^{-1} \left(\frac{1}{2} \right)$$

QUESTION 3 (Start a new page)

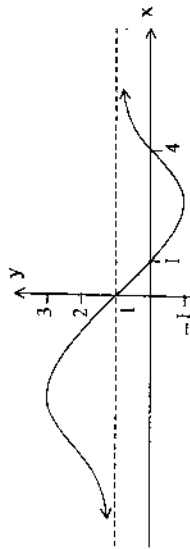
MARKS

- a) Consider the equation $x^3 + 7x - 6i = 0$.
- Given that this equation has no purely real root, show that none of the roots is a conjugate of any of the others.
 - If $2i$ is one of the roots and the other two roots are purely imaginary, find the other two roots.

1

2

b)



The above diagram shows the graph of $y = f(x)$. Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes.

- $y = \frac{1}{f(x)}$
- $y = f(|x|)$
- $y = \ln f(x)$
- $y = \sin^{-1}(f(x))$

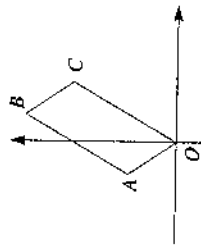
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c)



In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60^\circ$, what complex number does C represent?

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QUESTION 4 (Start a new page)

MARKS

- a) Factorise $P(x) = x^4 - 5x^3 + 4x^2 + 2x - 8$ over
- $\mathbb{R}$ (all real numbers)
 - $\mathbb{C}$ (all complex numbers)

3

- b) Write down all polynomials that have degree 4, 3 as a single zero and -1 as a zero of multiplicity 3.

1

- c) If α, β, ρ are the roots of $x^3 - 2x^2 + x + 3 = 0$ evaluate:

3

- $\alpha^2 + \beta^2 + \rho^2$
- $\alpha^3 + \beta^3 + \rho^3$

- d) If α, β, ρ are the roots of $x^3 + 2x^2 - 2x + 3 = 0$ form the equation whose roots are:

4

- $2\alpha, 2\beta, 2\rho$
- $\alpha^2, \beta^2, \rho^2$

- e) The roots of the polynomial $P(x) = x^3 + ax^2 + bx + c = 0$ are in arithmetic progression. Show that the relationship between the coefficients of $P(x)$ is $2a^3 = 9ab - 27c$

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- f) Prove that if α is a root of multiplicity r of $P(x)$ then it is a root of multiplicity $(r-1)$ of $P'(x)$.

1

QUESTION 5 (Start a new page)

MARKS

- a) i) Show that the equation of the chord of contact of the tangents from a point (x_0, y_0) to the rectangular hyperbola $xy = c^2$ is $xy_0 + x_0y = 2c^2$.
- ii) Hence find the chord of contact of the tangents from the point $(2, 1)$ to the hyperbola $xy = 4$ and determine the points of contact.

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- b) i) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2 m^2 + b^2$.
- ii) Hence show that the pair of tangents from the point $(3, 4)$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to one another.

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- c) i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a \sec \theta, b \tan \theta)$ is $a \sin \theta x + by = (a^2 + b^2) \tan \theta$.

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- ii) The normal at the point $P(a \sec \theta, b \tan \theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x-axis at G.

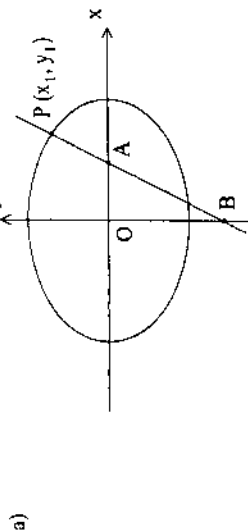
PN is the perpendicular from P to the x-axis.

Prove that $OG = e^2 \cdot ON$, where O is the origin.

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QUESTION 6 (Start a new page)

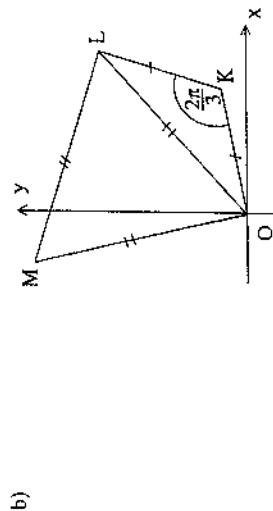
MARKS



a)

The point $P(x_1, y_1)$, where $x_1 > 0$ and $y_1 > 0$, lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The normal at P intersects the x-axis at A and the y-axis at B.

- i) Show that the equation of the normal is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$ 3
- ii) Explain why the point A cannot be the focus of the ellipse. 2
- iii) Find the ratio in which A divides the interval BP internally. 2
- iv) Find the midpoint M of AB in terms of x_1 and y_1 . 1
- v) Given that H divides the interval OM in the ratio 4:1, show that the locus of H is an ellipse. 3



b)

The points K and M in a complex plane represent the complex numbers α and β respectively. The triangle OKL is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle OLM is equilateral.

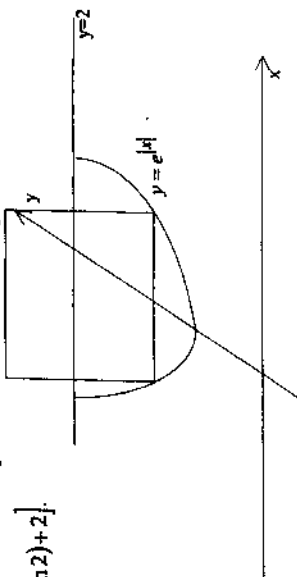
Show that $3\alpha^2 + \beta^2 = 0$

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QUESTION 7 (Start a new page)

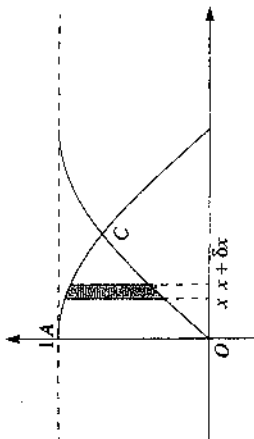
MARKS

- a) The base of a solid is formed by the segment cut off by the line $y = 2$ of the curve $y = e^{|x|}$. Cross sections of the solid parallel to the x axis are squares. Show that the volume is given by $4[2(\ln 2)^2 - 4(\ln 2) + 2]$.



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- b) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y axis at A, and the C is the first point of intersection of the two graphs to the right of the y axis.



The region OAC is to be rotated about the line $y = 1$.

- Write down the coordinates of the point C. 1
- The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by $\delta V = \pi(2 \cos x - 2 \sin x + \sin^2 x - \cos^2 x) \delta x$. 3
- Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$. 3

QUESTION 8 (Start a new page)

MARKS

- a) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$, where n is a positive integer.

i) Using integration, show that

$$(n-1) I_n = 2^{n-2} \sqrt{3} + (n-2) I_{n-2}$$

4

ii) Evaluate $J = \int_0^{\frac{\pi}{3}} \sec^4 x \, dx$

3

b)

Consider the polynomial $x^5 - i = 0$

i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$

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ii) Show that

$$(x-i) \left(x^2 - 2i \sin \frac{\pi}{10} x - 1 \right) \left(x^2 + 2i \sin \frac{3\pi}{10} x - 1 \right) = 0$$

4

iii) Show that $\sin \frac{\pi}{10} \sin \frac{3\pi}{10} = \frac{1}{4}$

2