CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2001
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2
Marking guidelines/ solutions

2604 - 2

Question 1

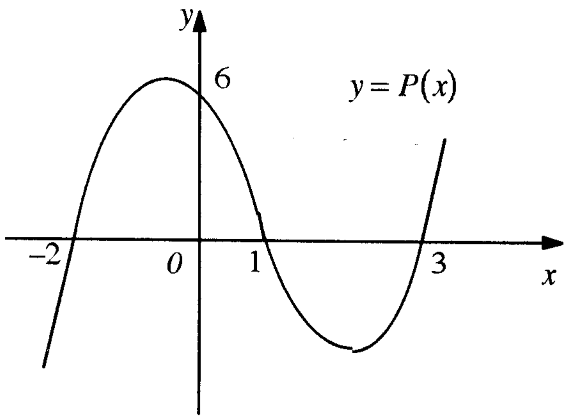
(a) **Outcomes Assessed:** (i) **PE3** (ii) **E6**

Marking Guidelines

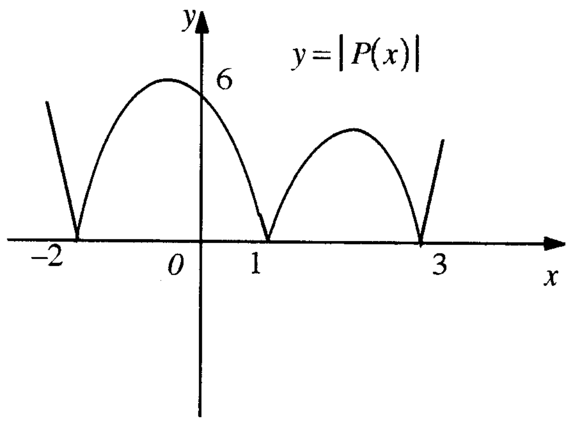
Criteria	Marks
(i) • one mark for graph of $y = P(x)$	1
(ii) • one mark for graph of $y = \lvert P(x) \rvert$ • one mark for graph of $y = P(\lvert x \rvert)$ • one mark for asymptotes and intercepts of graph of $y = \dfrac{1}{P(x)}$ • one mark for graph of $y = \dfrac{1}{P(x)}$	4

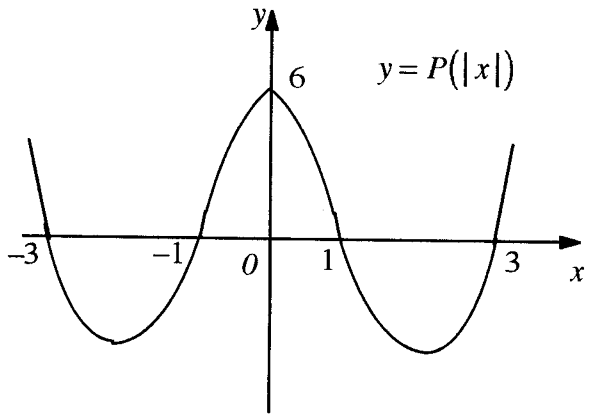
Answer $P(x) = (x+2)(x-1)(x-3)$

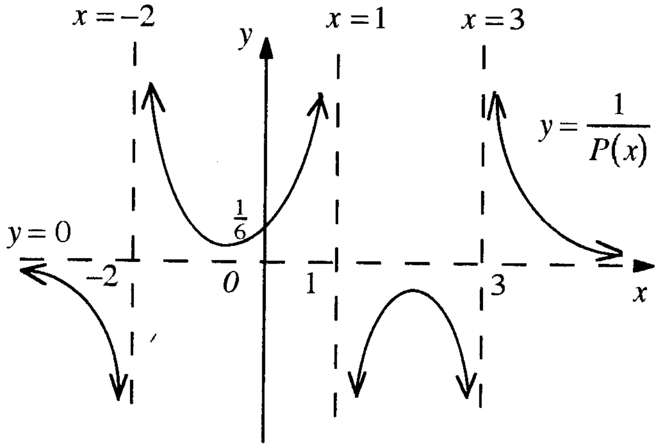
(i)



(ii)





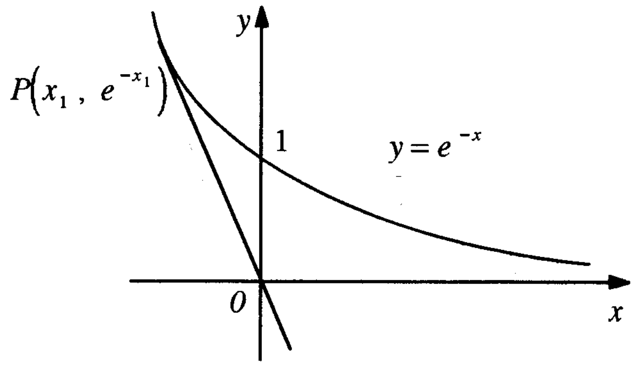


(b) **Outcomes Assessed:** (i) **E6** (ii) **E6**

Marking Guidelines

Criteria	Marks
(i) • one mark for gradient $OP = \dfrac{e^{-x_1}}{x_1}$ • one mark for gradient $OP = -e^{-x_1}$ • one mark for coordinates of P	3
(ii) • one mark for gradient of tangent $= -e$ • one mark for set of values of k	2

Answer



(i) $P(x_1, e^{-x_1})$

$y=e^{-x}$

$\frac{dy}{dx}=-e^{-x}$

$grad.\ OP = \frac{e^{-x_1}}{x_1}$

$grad.\ tangent\ at\ P = -e^{-x_1}$

Since OP is tangent at P, $\frac{e^{-x_1}}{x_1} = -e^{-x_1}$

$\therefore (x_1+1)e^{-x_1}=0$

$\therefore x_1=-1$, $P(-1, e)$

(ii) $y=-e\,x$ is tangent to the curve $y=e^{-x}$ at $P(-1, e)$, and intersects the curve at no other point.
By inspection of the graph, for $-e<k\le 0$, $y=kx$ has no points of intersection with the curve.
for $k>0$, $y=kx$ has exactly one point of intersection with the curve.
Since $y=e^{-x}$ is steeper than any linear function of x as $x\to-\infty$, lines $y=kx$, $k<-e$, will intersect the curve in two distinct points.
Hence $e^{-x}=kx$ has two real and distinct solutions for $\{k: k<-e\}$.

(c) **Outcomes Assessed**: (i) **P5, H5** (ii) **E6**

Marking Guidelines

Criteria	Marks
(i) • one mark for showing $f(-x)=f(x)$ • one mark for finding $f''(x)$ • one mark for showing $f''(x)<0$	3
(ii) • one mark for asymptotes, endpoints and intercepts of graph $y=f(x)$ • one mark for graph $y=f(x)$	2

Answer

(i)

$f(x)=\ln(1+\cos x)$

$f(-x)=\ln\{1+\cos(-x)\}=\ln(1+\cos x)=f(x)$

Hence f is an even function.

$$f'(x) = \frac{-\sin x}{1+\cos x}$$

$$f''(x) = -\frac{\cos x(1+\cos x)-\sin x(-\sin x)}{(1+\cos x)^2}$$

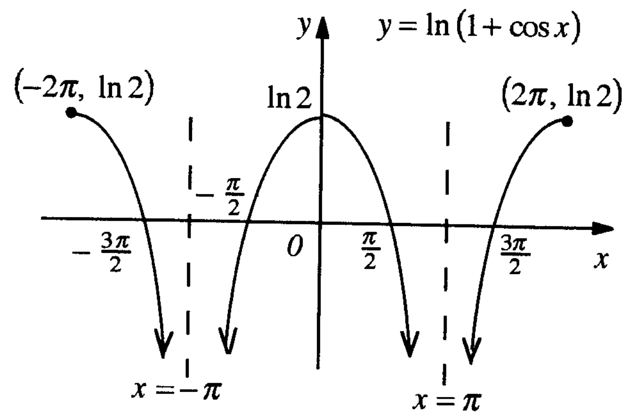
$$= -\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$= -\frac{\cos x+1}{(1+\cos x)^2}$$

$$\therefore f''(x)=\frac{-1}{1+\cos x}<0 \quad (\text{since } 1+\cos x>0, x\neq\pm\pi\ )$$

Hence curve is concave down throughout its domain.

(ii)



Question 2

(a) **Outcomes Assessed: E3**

Marking Guidelines

Criteria	Marks
• one mark for equating imaginary parts to evaluate a • one mark for equating real parts to get equation in b • one mark for values of z	3

Answer

$z = a + ib$, a, b real.

$|z|^2 - iz = a^2 + b^2 - ia + b$

$\therefore\ 16 - 2i = (a^2 + b^2 + b) - ia$

Equating real and imaginary parts,

$$\left.\begin{aligned} a &= 2 \\ a^2 + b^2 + b &= 16 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} b^2 + b - 12 &= 0 \\ (b+4)(b-3) &= 0 \end{aligned}$$

$\therefore\ a = 2,\ b = -4$ or $a = 2,\ b = 3$

Hence $z = 2 - 4i$ or $z = 2 + 3i$

(b) **Outcomes Assessed:** (i) **H5** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integration	1
(ii) • one mark for partial fractions • one mark for integration	2

Answer

(i) $\displaystyle\int \frac{e^x + 1}{e^x}\,dx = \int \left(1 + e^{-x}\right)\,dx = x - e^{-x} + c$

(ii) $\displaystyle\int \frac{x^2 + x + 1}{x(x^2+1)}\,dx = \int \frac{(x^2+1) + x}{x(x^2+1)}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2+1}\right)\,dx = \ln|x| + \tan^{-1} x + c$

(c) **Outcomes Assessed:** (i) **E8** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integral in terms of t • one mark for evaluation of integral	2
(ii) • one mark for integral in terms of u • one mark for evaluation of integral	2

Answer

(i)

$t = \tan \frac{x}{2}$

$dt = \frac{1}{2}\sec^2 \frac{x}{2}\,dx$

$2\,dt = \left(1 + \tan^2 \frac{x}{2}\right)dx$

$dx = \dfrac{2}{1+t^2}\,dt$

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$1 + \cos x + \sin x = 1 + \dfrac{1-t^2}{1+t^2} + \dfrac{2t}{1+t^2} = \dfrac{2+2t}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x + \sin x}\,dx = \int_0^1 \frac{1+t^2}{2(1+t)} \cdot \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{1}{1+t}\,dt$$

$$= \Big[\ln|1+t|\Big]_0^1$$

$$= \ln 2$$

(ii) Let $I=\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$

$u=\frac{\pi}{2}-x$

$du=-dx$

$x=0 \Rightarrow u=\frac{\pi}{2}$

$x=\frac{\pi}{2} \Rightarrow u=0$

$x=\frac{\pi}{2}-u$

$\cos x+\sin x=\sin u+\cos u$

$$I = \int_{\frac{\pi}{2}}^{0} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u}\cdot -du = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}-u}{1+\cos u+\sin u}\,du$$

$$\therefore I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos u+\sin u}\,du - \int_0^{\frac{\pi}{2}} \frac{u}{1+\cos u+\sin u}\,du$$

$$I = \frac{\pi}{2}\ln 2 - I$$

$$2I = \frac{\pi}{2}\ln 2$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx = \frac{\pi}{4}\ln 2$$

(d) **Outcomes Assessed:** (i) **E8** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integration by parts • one mark for use of $x^2=(1+x^2)-1$ • one mark for obtaining recurrence relation	3
(ii) • one mark for integral in terms of $u=\tan x$ • one mark for recurrence relation	2

Answer

(i)

$$I_n=\int_0^1 (1+x^2)^n\,dx$$

$$=\left[x\,(1+x^2)^n\right]_0^1-\int_0^1 x\,.\,n\,(1+x^2)^{n-1}.2x\,dx$$

$$=2^n - 2n\int_0^1 x^2(1+x^2)^{n-1}\,dx$$

$$=2^n-2n\int_0^1 (1+x^2-1)(1+x^2)^{n-1}\,dx$$

$$=2^n-2n\left\{\int_0^1 (1+x^2)^n\,dx-\int_0^1 (1+x^2)^{n-1}\,dx\right\}$$

$$I_n=2^n-2n\,I_n + 2n\,I_{n-1}$$

$$\therefore (2n+1)\,I_n=2^n+2n\,I_{n-1}\ ,\quad n=1,2,3,\ldots$$

(ii)

$u=\tan x$ $\qquad x=0 \Rightarrow u=0$

$du=\sec^2 x\,dx$ $\qquad x=\frac{\pi}{4} \Rightarrow u=1$

$$J_m=\int_0^{\frac{\pi}{4}} \sec^{2m} x\,dx$$

$$=\int_0^{\frac{\pi}{4}} (\sec^2 x)^{m-1}.\ \sec^2 x\,dx$$

$$=\int_0^1 (1+u^2)^{m-1}\,du$$

$$\therefore\ J_m=I_{m-1}\ ,\quad m=1,2,3,\ldots$$

$$\{2(m-1)+1\}J_m=2^{m-1}+2(m-1)\,I_{m-2}$$

$$\therefore\ (2m-1)\,J_m=2^{m-1}+2(m-1)\,J_{m-1}$$

$m=2,\ 3,\ 4,\ldots$

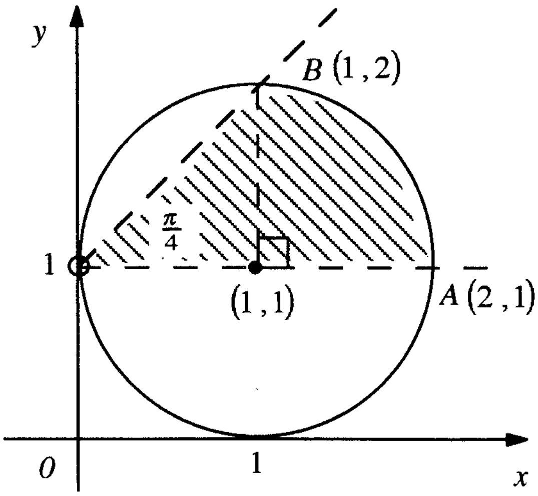
Question 3

(a) **Outcomes Assessed:** (i)**E3** (ii) **E3**

Marking Guidelines

Criteria	Marks
(i) • one mark for sketch	1
(ii) • one mark for shading region	1

Answer (i), (ii) Locus of P is the circle centred on $(1, 1)$ with radius 1 unit.

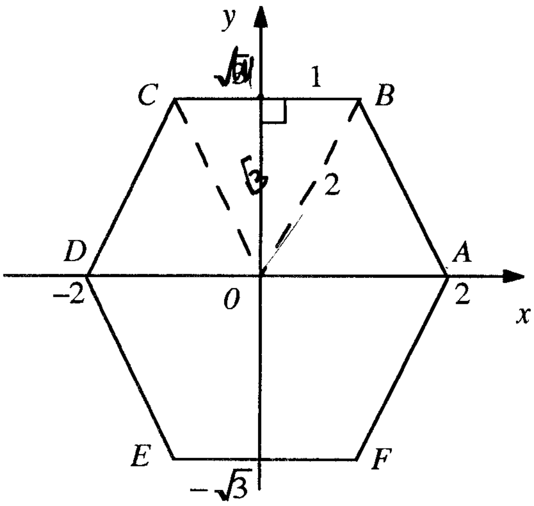


b) **Outcomes Assessed:** (i) **E3** (ii) **E3** (iii) **E3**

Marking Guidelines

Criteria	Marks
(i) • one mark for set of values of $\text{Im}(z)$	1
(ii) • one mark for set of values of $\lvert z \rvert$	1
(iii) • one mark for value of complex number	1

Answer



(i) $-\sqrt{3} \le \text{Im}(z) \le \sqrt{3}$

(ii) $\sqrt{3} \le |z| \le 2$

(iii) Each of the triangles ΔAOB, $\Delta BOC, \ldots$ is equilateral with side 2 units.

$\therefore\ A\hat{O}C = 2 \times 60° = 120°$

After rotation clockwise through $45°$, OC will make an angle $75°$, or $\frac{5\pi}{12}$ radians, with the positive x axis. Hence C will then represent the complex number $2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$.

c) **Outcomes Assessed:** (i) **E2, E3** (ii) **E2, E3** (iii) **E4** (iv) **E4**

Marking Guidelines

Criteria	Marks
(i) • one mark for use of De Moivre's Theorem to obtain expressions for $z^n \pm \dfrac{1}{z^n}$ • one mark for expansion of $\left(z+\frac{1}{z}\right)^4 + \left(z-\frac{1}{z}\right)^4$ in terms of z • one mark for obtaining expression for $\cos^4\theta + \sin^4\theta$ in terms of $\cos 4\theta$	3
(ii) • one mark for showing equation reduces to $\cos 4\theta = \frac{1}{2}$ • one mark for solving this equation to obtain values of x	2
(iii) • one mark for using product of roots in terms of coefficients to evaluate $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}$ • one mark for using sum of products of roots taken two at a time in terms of coefficients to evaluate $\cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12}$ • one mark for evaluating $\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}$	3
(iv) • one mark for forming quadratic equation with roots $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$ • one mark for value of $\cos\frac{\pi}{12}$	2

Answer

(i) Using De Moivre's Theorem,

$z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$

$\therefore\ z^n + \dfrac{1}{z^n} = 2\cos n\theta,\quad z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

$$\left(z+\frac{1}{z}\right)^4 + \left(z-\frac{1}{z}\right)^4 = 2\left(z^4 + 6z^2\cdot\frac{1}{z^2} + \frac{1}{z^4}\right)$$

$$= 2\left(z^4 + \frac{1}{z^4}\right) + 12$$

$$(2\cos\theta)^4 + (2i\sin\theta)^4 = 2(2\cos 4\theta) + 12$$

$$16(\cos^4\theta + \sin^4\theta) = 4(\cos 4\theta + 3)$$

$$\therefore\quad \cos^4\theta + \sin^4\theta = \tfrac{1}{4}(\cos 4\theta + 3)$$

(ii)

$x = \cos\theta,\qquad 8x^4 + 8(1-x^2)^2 = 7$

$1 - x^2 = \sin^2\theta \Rightarrow 8(\cos^4\theta + \sin^4\theta) = 7$

$2(\cos 4\theta + 3) = 7$

Hence equation becomes

$x = \cos\theta,\qquad \cos 4\theta = \tfrac{1}{2}$

$$4\theta = 2n\pi \pm \tfrac{\pi}{3} \Rightarrow \theta = \frac{(6n\pm1)\pi}{12}$$

$n = 0, \pm1, \pm2, \ldots$

$x = \cos\frac{\pi}{12},\ \cos\frac{5\pi}{12},\ \cos\frac{7\pi}{12},\ \cos\frac{11\pi}{12}$

$x = \cos\frac{\pi}{12},\ \cos\frac{5\pi}{12},\ \cos\left(\pi - \frac{5\pi}{12}\right),\ \cos\left(\pi - \frac{\pi}{12}\right)$

$\therefore\ x = \pm\cos\frac{\pi}{12},\ \pm\cos\frac{5\pi}{12}$

(iii) $8x^4 + 8(1-x^2)^2 = 7$ simplifies to give

$16x^4 - 16x^2 + 1 = 0$,

with roots $\cos\frac{\pi}{12},\ -\cos\frac{\pi}{12},\ \cos\frac{5\pi}{12},\ -\cos\frac{5\pi}{12}$.

Then $\alpha\beta\gamma\delta = \cos^2\frac{\pi}{12}\ \cos^2\frac{5\pi}{12} = \frac{1}{16}$

$\sum\alpha\beta = -\cos^2\frac{\pi}{12} - \cos^2\frac{5\pi}{12} = -1$

where $0 < \frac{\pi}{12} < \frac{5\pi}{12} < \frac{\pi}{2}$.

Then $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = +\sqrt{\frac{1}{16}} = \frac{1}{4}$, and

$\cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + 2\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = 1 + \frac{1}{2}$

$\therefore\quad \left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right)^2 = \frac{3}{2}$

$\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \sqrt{\frac{3}{2}}$

(iv) $\cos\frac{\pi}{12},\ \cos\frac{5\pi}{12}$ are roots of the quadratic equation $x^2 - \sqrt{\frac{3}{2}}\,x + \frac{1}{4} = 0$.

$$x = \frac{\sqrt{\frac{3}{2}} \pm \sqrt{\frac{3}{2} - 1}}{2} = \frac{\sqrt{3}\pm1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{12} > \cos\frac{5\pi}{12} \Rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Question 4

Outcomes Assessed: (i) **E2, E3, E4** (ii) **E2, E3, E4** (iii) **E2, E4** (iv) **E2, E4** (v) **E4, E6** (vi) **E2, E4, E9** (vii) **E2, E4, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for finding gradient of tangent in terms of t • one mark for obtaining equation of tangent	2
(ii) • one mark for finding gradient of tangent in terms of θ • one mark for finding equation of tangent	2
(iii) • one mark for comparing coefficients to obtain result	1
(iv) • one mark for coordinates of Q in terms of t • one mark for obtaining quartic equation in t • one mark for using this equation to deduce there are exactly two common tangents	3
(v) • one mark for diagram showing second common tangent • one mark for coordinates of R and S	2
(vi) • one mark for using geometrical properties of a rhombus to show $b^2 = a^2$ • one mark for deducing $t^2 < 1$	2
(vii) • one mark for using geometrical properties of a square to obtain equation in t • one mark for deducing that $2c^2 = a^2$ • one mark for recognising the relationship between the rectangular hyperbolas	3

Answer

(i)

$$x = ct \Rightarrow \frac{dx}{dt} = c, \quad y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = \frac{-c}{t^2} \quad \therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{1}{t^2}$$

Hence tangent l has gradient $-\frac{1}{t^2}$ and equation $x + t^2 y = k$, k constant, where $P\left(ct, \frac{c}{t}\right)$ lies on $l \Rightarrow ct + ct = k$. Hence l has equation $x + t^2 y = 2ct$.

(ii)

$$x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{b\sec\theta}{a\tan\theta}$$

Hence tangent l has gradient $\frac{b\sec\theta}{a\tan\theta}$ and equation $x\,b\sec\theta - y\,a\tan\theta = k$, k constant, where $Q(a\sec\theta, b\tan\theta)$ lies on l $\Rightarrow k = ab\sec^2\theta - ab\tan^2\theta = ab$. Hence l has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

(iii) Comparing the two forms of the equation of line l, the coefficients must be in proportion. Hence

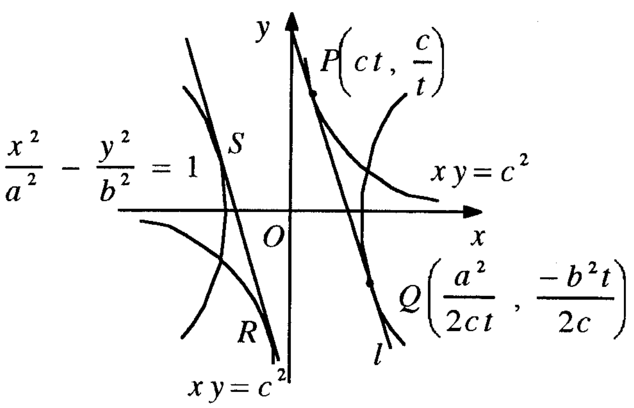
$$\frac{\left(\frac{\sec\theta}{a}\right)}{1} = \frac{\left(\frac{-\tan\theta}{b}\right)}{t^2} = \frac{1}{2ct} \qquad \therefore \frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}.$$

(iv)

$$\left.\begin{array}{l} Q(a\sec\theta, b\tan\theta) \\ \equiv Q\left(\frac{a^2}{2ct}, \frac{-b^2t}{2c}\right) \end{array}\right\}, \quad \left.\begin{array}{l} \sec^2\theta - \tan^2\theta = 1 \\ \left(\frac{a}{2ct}\right)^2 - \left(\frac{-bt}{2c}\right)^2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} a^2 - b^2t^4 = 4c^2t^2 \\ b^2t^4 + 4c^2t^2 - a^2 = 0 \end{array}$$

This quadratic in t^2 has discriminant $\Delta = 16c^4 + 4a^2b^2 > 0$, and hence has two real roots which are opposite in sign (since their product is negative). But $t^2 > 0$, hence there is exactly one solution for t^2, and two solutions for t which are opposites of each other. Each such value of t gives a common tangent l to the two hyperbolas.

(v)



$$R\left(-ct, \frac{-c}{t}\right), \quad S\left(\frac{-a^2}{2ct}, \frac{b^2t}{2c}\right)$$

(vi)

O is the common midpoint of diagonals PR and QS. Hence $PQRS$ is a parallelogram.

$$\textit{gradient } PR = \frac{2c}{t} \div 2ct = \frac{1}{t^2}$$

$$\textit{gradient } QS = \frac{b^2t}{c} \div \frac{-a^2}{ct} = \frac{b^2}{a^2}(-t^2)$$

$$\therefore \textit{gradient } PR \,.\, \textit{gradient } QS = -\frac{b^2}{a^2}$$

Hence if $PQRS$ is a rhombus, $PR \perp QS$ and *gradient PR* . *gradient QS* $= -1 \Rightarrow b^2 = a^2$.

Then t satisfies $a^2t^4 + 4c^2t^2 - a^2 = 0$

$$t^4 + \frac{4c^2}{a^2}t^2 = 1$$

$$\left(t^2 + \frac{2c^2}{a^2}\right)^2 = 1 + \frac{4c^4}{a^4} < \left(1 + \frac{2c^2}{a^2}\right)^2$$

Hence $t^2 < 1$

(vii) If $PQRS$ is a square, then $PQRS$ is a rhombus with $R\hat{P}Q = 45°$. Then

$$\left.\begin{array}{l} \textit{gradient } PR = \dfrac{1}{t^2} \\ \textit{gradient } PQ = \dfrac{-1}{t^2} \end{array}\right\} \Rightarrow 1 = \left| \dfrac{\left(\dfrac{2}{t^2}\right)}{1+\left(\dfrac{1}{t^2}\right)\left(\dfrac{-1}{t^2}\right)} \right| = \dfrac{-2t^2}{t^4-1} \quad \text{(since } t^2<1 \text{ for } PQRS \text{ a rhombus)}$$

Hence $t^4 + 2t^2 - 1 = 0$. But for $PQRS$ a rhombus, t satisfies $t^4 + \dfrac{4c^2}{a^2}t^2 - 1 = 0$.

By subtraction, $\left(\dfrac{4c^2}{a^2} - 2\right)t^2 = 0$. But $t^2 \neq 0$. Hence $2c^2 = a^2$.

Hence if $PQRS$ is a square (and hence a rhombus), then $b^2 = a^2$, and the two hyperbolas have equations $x^2 - y^2 = a^2$ and $xy = c^2$, where $2c^2 = a^2$.

This relationship between c^2 and a^2 means that the rectangular hyperbola $x^2 - y^2 = a^2$ rotated anticlockwise through $45°$ becomes the rectangular hyperbola $xy = c^2$.

Question 5

(a) **Outcomes Assessed**: (i) **E8** (ii) **H5** (iii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for integration by parts of $I-J$ • one mark for obtaining result	2
(ii) • one mark for finding $\int (x+1)e^x \, dx$ from the derivative of xe^x • one mark for finding the required expression for $I+J$	2
(iii) • one mark for value of I	1

Answer

(i) $I = \displaystyle\int_0^{\pi} x e^x \cos x \, dx, \quad J = \int_0^{\pi} e^x \cos x \, dx$

$$I - J = \int_0^{\pi} (x-1) e^x \cos x \, dx$$

$$= \left[(x-1) e^x \sin x\right]_0^{\pi} - \int_0^{\pi} x e^x \sin x \, dx$$

$$= -\int_0^{\pi} x e^x \sin x \, dx$$

(ii) $\dfrac{d}{dx} xe^x = e^x + xe^x = (x+1)e^x$

$$\therefore \int (x+1)e^x \, dx = xe^x + c$$

$$I + J = \int_0^{\pi} (x+1) e^x \cos x \, dx$$

$$= \left[x e^x \cos x\right]_0^{\pi} - \int_0^{\pi} x e^x(-\sin x) \, dx$$

$$= -\pi e^{\pi} + \int_0^{\pi} x e^x \sin x \, dx$$

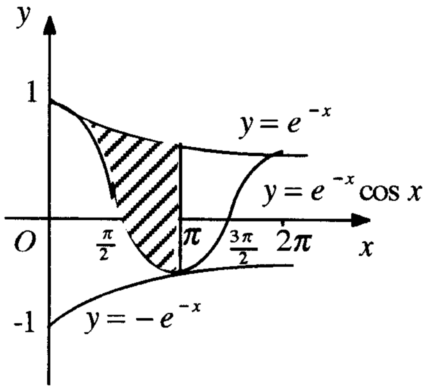
(iii) $I = \frac{1}{2}\{(I+J)+(I-J)\} = -\frac{1}{2}\pi e^{\pi}$

(b) **Outcomes Assessed**: (i) **E6** (ii) **E7** (iii) **E8** (iv) **E8**

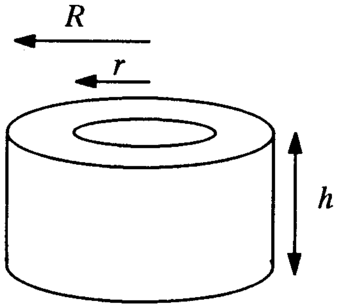
Marking Guidelines

Criteria	Marks
(i) • one mark for graphs of $y=e^{-x}$, $y=-e^{-x}$ • one mark for graph of $y=e^{-x}\cos x$ • one mark for shading region	3
(ii) • one mark for expression for volume of cylindrical shell δV in terms of x • one mark for using concept of limiting sum to form integral for V	2
(iii) • one mark for expressing integral for V in terms of $u=\pi-x$ • one mark for rearrangement to express V in terms of I	2
(iv) • one mark for integration by parts for $\int ue^{-u}\,du$ • one mark for evaluation of $\int ue^{-u}\,du$ • one mark for evaluating V	3

(i)



(ii)



$R=\pi-x+\delta x$, $r=\pi-x$

$h=e^{-x}-e^{-x}\cos x$

Cylindrical shell has volume

$\delta V=\pi\left(R^2-r^2\right)e^{-x}(1-\cos x)$

where

$$\begin{aligned}R^2-r^2&=(R+r)(R-r)\\&=\{2(\pi-x)+\delta x\}\delta x\\&=2(\pi-x)\,\delta x\end{aligned}$$

ignoring terms in $(\delta x)^2$.

Hence volume of solid of revolution is given by

$$V=\lim_{\delta x\to 0}\sum_{x=0}^{x=\pi}\delta V=2\pi\int_0^{\pi}(\pi-x)e^{-x}(1-\cos x)\,dx.$$

(iii)

$u=\pi-x$ $du=-dx$

$x=0\Rightarrow u=\pi$

$x=\pi\Rightarrow u=0$

$$\begin{aligned}1-\cos x&=1-\cos(\pi-u)\\&=1+\cos u\end{aligned}$$

$$\begin{aligned}V&=2\pi\int_{\pi}^{0}u\,e^{u-\pi}\{1+\cos u\}\,(-du)\\&=2\pi\,e^{-\pi}\int_0^{\pi}u\,e^{u}\{1+\cos u\}\,du\\&=2\pi\,e^{-\pi}\left\{\int_0^{\pi}u\,e^{u}\,du+\int_0^{\pi}u\,e^{u}\cos u\,du\right\}\\&=2\pi\,e^{-\pi}\left\{\int_0^{\pi}u\,e^{u}\,du+I\right\}\end{aligned}$$

(iv)

$$\begin{aligned}\int_0^{\pi}ue^{u}\,du&=\left[ue^{u}\right]_0^{\pi}-\int_0^{\pi}e^{u}\,du\\&=\pi e^{\pi}-\left[e^{u}\right]_0^{\pi}\\&=\pi e^{\pi}-\left(e^{\pi}-1\right)\end{aligned}$$

$$\begin{aligned}V&=2\pi e^{-\pi}\left\{\pi e^{\pi}-e^{\pi}+1+I\right\}\\&=2\pi e^{-\pi}\left\{\pi e^{\pi}-e^{\pi}+1-\tfrac{1}{2}\pi e^{\pi}\right\}\\&=\pi(\pi-2)+2\pi e^{-\pi}\end{aligned}$$

Hence volume is $\pi(\pi-2)+2\pi e^{-\pi}$ cu. units.

Question 6

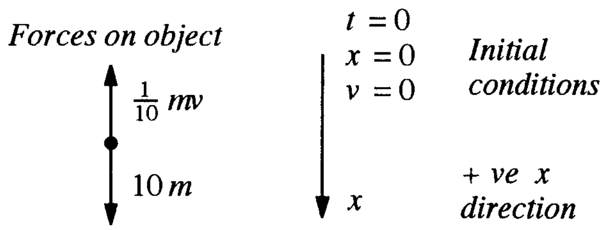
(a) **Outcomes Assessed:** (i) **E2, E5** (ii) **E2, E5** (iii) **PE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for $\ddot{x}$ in terms of v	1
(ii) • one mark for obtaining expression for $\dfrac{dv}{dx}$ • one mark for integration using initial conditions to find expression for x in terms of v • one mark for obtaining required equation for speed V on entry to water	3
(iii) • one mark for showing there is a solution for V lying between 20 and 30 • one mark for applying Newton's method to find expression for next approximation • one mark for obtaining value of V	3

Answer

(i)



$m\ddot{x} = 10m - \frac{1}{10}mv \qquad \therefore \ \ddot{x} = 10 - \frac{1}{10}v$

(ii) $\ddot{x} = v\dfrac{dv}{dx} = 10 - \frac{1}{10}v \ \Rightarrow\ 10\dfrac{dv}{dx} = \dfrac{100-v}{v}$

$\dfrac{-1}{10}\dfrac{dx}{dv} = \dfrac{-v}{100-v} = 1 + \dfrac{-100}{100-v}$

$-\frac{1}{10}x = v + 100\ln(100-v) + c$, c constant

$t=0,\ x=0,\ v=0 \ \Rightarrow\ c = -100\ \ln 100$

$\therefore\ -\frac{1}{10}x = v + 100\ln\left(1 - \frac{v}{100}\right)$

$\left.\begin{array}{l} x=40 \\ v=V \end{array}\right\} \Rightarrow \begin{array}{l} -4 = V + 100\ln\left(1-\frac{V}{100}\right) \\ -0\cdot04 = \frac{V}{100} + \ln\left(1-\frac{V}{100}\right) \end{array}$

$\therefore$ Speed V ms^{-1} just before entering water satisfies

$\frac{V}{100} + \ln\left(1-\frac{V}{100}\right) + 0\cdot04 = 0 \qquad$ **

(iii)

Let $\lambda = \frac{V}{100}$, $f(\lambda) = \lambda + \ln(1-\lambda) + 0\cdot04$

$f(0\cdot2) \approx 0\cdot02 > 0 \qquad f(0\cdot3) \approx -0\cdot02 < 0$

and $f(\lambda)$ is a continuous function. Hence $f(\lambda)=0$ has a solution for λ between $0\cdot2$ and $0\cdot3$, and ** has a solution for V between 20 and 30. Using Newton's Method with a first approximation $\lambda = 0\cdot25$ $(V=25)$

$$f(\lambda) = \lambda + \ln(1-\lambda) + 0\cdot04$$

$$f'(\lambda) = 1 - \frac{1}{1-\lambda} = \frac{-\lambda}{1-\lambda}$$

$$\frac{f(\lambda)}{f'(\lambda)} = \{\lambda + \ln(1-\lambda) + 0\cdot04\}\left(\frac{1-\lambda}{-\lambda}\right)$$

$$= \lambda - 1 - \frac{(1-\lambda)\{\ln(1-\lambda)+0\cdot04\}}{\lambda}$$

$$\lambda - \frac{f(\lambda)}{f'(\lambda)} = 1 + \frac{(1-\lambda)\{\ln(1-\lambda)+0\cdot04\}}{\lambda}$$

λ	$1+\frac{1-\lambda}{\lambda}\{\ln(1-\lambda)+0\cdot04\}$
$0\cdot25$	$1+3(\ln 0\cdot75 + 0\cdot04) = 0\cdot257$
$0\cdot257$	$1+\frac{0\cdot743}{0\cdot257}(\ln 0\cdot743 + 0\cdot04) = 0\cdot257$

Hence $\lambda = 0\cdot257 \Rightarrow V = 25\cdot7$ to one decimal place.

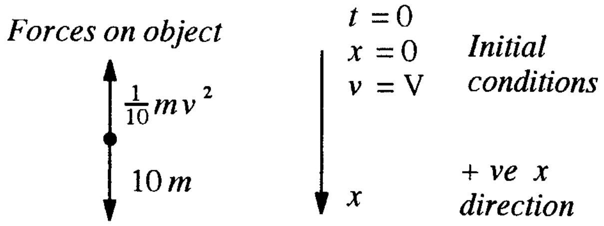
(b) **Outcomes Assessed**: (i) **E2, E5** (ii) **E2, E5** (iii) **E5**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for $\ddot{x}$ in terms of v • one mark for deducing object slows on entry to water • one mark for finding terminal velocity	3
(ii) • one mark for obtaining expression for $\frac{dv}{dt}$ • one mark for expressing $\frac{dv}{dt}$ in terms of partial fractions • one mark for integration using initial conditions to find expression for t in terms of v	3
(iii) • one mark for selecting correct value of v to substitute in expression for t • one mark for value of t	2

Answer

(i) After entering the water,



$m\ddot{x} = 10m - \frac{1}{10}mv^2 \qquad \therefore\ \ddot{x} = 10 - \frac{1}{10}v^2$

$\ddot{x} = 10 - \frac{1}{10}V^2 < 0$ and $\dot{x} = V > 0$
Hence object slows on entry to the water.

$\ddot{x} \to 0$ as $v \to 10$
Hence terminal velocity in the water is $10\ \text{ms}^{-1}$.

(ii) $\ddot{x} = \dfrac{dv}{dt} = 10 - \frac{1}{10}v^2 \Rightarrow 10\dfrac{dv}{dt} = 100 - v^2$

$$\frac{1}{10}\frac{dt}{dv} = \frac{1}{(10+v)(10-v)}$$

$$= \frac{1}{20}\left\{\frac{1}{(10+v)} + \frac{1}{(10-v)}\right\}$$

$$2\frac{dt}{dv} = \frac{1}{(v+10)} - \frac{1}{(v-10)}$$

$$2t = \ln\left\{\frac{(v+10)}{(v-10)}A\right\}, \quad A \text{ constant}$$

$$\left.\begin{matrix} t=0 \\ v=V \end{matrix}\right\} \Rightarrow \frac{(V+10)}{(V-10)}A = 1 \Rightarrow A = \frac{(V-10)}{(V+10)}$$

$$\therefore\ 2t = \ln\left\{\frac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$$

(iii) $v = 105\%$ of $10 \Rightarrow v = 10\cdot5$ and $2t \approx \ln\left\{\dfrac{(20\cdot5)(15\cdot7)}{(0\cdot5)(35\cdot7)}\right\} \Rightarrow t \approx 1\cdot4$.

Hence particle slows to 105% of its terminal velocity $1\cdot4$ seconds after entering the water.

Question 7

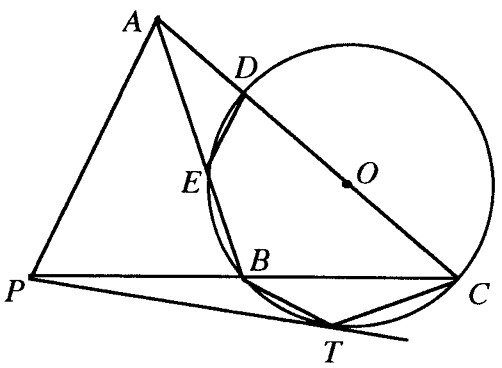
(a) **Outcomes Assessed**: (ii) **PE2, PE3, E2, E9** (iii) **PE2, PE3, E2, E9** (iv) **PE2, PE3, E2, E9**

Marking Guidelines

Criteria	Marks
(i) • no marks fo copying diagram	
(ii) • one mark for $\angle BTP = \angle TCP$ with reason • one mark for completing deduction of similarity with reasons	2
(iii) • one mark for $\frac{PB}{PT} = \frac{PT}{PC}$ with reason • one mark for $\frac{PB}{PA} = \frac{PA}{PC}$ with reason • one mark for completing deduction of similarity with reasons	3
(iv) • one mark for $\angle PAE = \angle BCD$ with reason • one mark for $\angle BCD = \angle DEA$ with reason • one mark for reason for $DE \parallel AP$	3

Answer

(i)



(ii) In ΔPBT, ΔPTC

$T\hat{P}B = C\hat{P}T$ (common angle)

$B\hat{T}P = T\hat{C}P$ (angle between chord BT and tangent PT is equal to angle in alternate segment)

$\therefore$ $\Delta PBT \parallel\!| \Delta PTC$ (two pairs of corresponding angles are equal)

(iii) In ΔAPB, ΔCPA

$\frac{PB}{PT} = \frac{PT}{PC}$ (corresponding sides of similar triangles ΔPBT, ΔPTC are in proportion)

$\therefore$ $\frac{PB}{PA} = \frac{PA}{PC}$ (given $PT = PA$)

$A\hat{P}B = C\hat{P}A$ (common angle)

$\therefore$ $\Delta APB \parallel\!| \Delta CPA$ (two pairs of corresponding sides in proportion and included angles are equal)

(iv) $P\hat{A}E = B\hat{C}D$ (corresponding angles of similar triangles ΔAPB, ΔCPA are equal)

$B\hat{C}D = D\hat{E}A$ (exterior angle of cyclic quadrilateral $BCDE$ is equal to interior opposite angle)

$\therefore$ $P\hat{A}E = D\hat{E}A$

$\therefore$ $DE \parallel AP$ (equal alternate angles on transversal AE)

(b) **Outcomes Assessed**: (i) **HE2, E2, E9** (ii) **H5, E2, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for showing statement $A(n)$: $u_n = 4^n - 2^n$ is true for $n = 1$, $n = 2$ • one mark for using reduction formula to express u_{k+1} in terms of expressions for u_k, u_{k-1} when $A(n)$ is true for $n \le k$ • one mark for concluding that if $A(n)$ is true for $n \le k$, then $A(k+1)$ is true • one mark for deducing that $A(n)$ is true for $n \ge 1$	4
(ii) • one mark for recognising S_n as partial sum of the difference of two geometric series • one mark for finding expression for S_n in terms of the individual partial sums • one mark for values of a, b, c	3

Answer

Let $A(n)$ be the statement: $u_n = 4^n - 2^n$, $n = 1, 2, 3, \ldots$

(i) *Consider* $A(1), A(2)$: $4^1 - 2^1 = 2 = u_1$, $4^2 - 2^2 = 12 = u_2$ $\quad\therefore A(1), A(2)$ are both true.

If $A(n)$ *is true for positive integers* $n \le k$ (k some positive integer, $k \ge 2$) , *then*

$$u_n = 4^n - 2^n, \quad n = 1, 2, 3, \ldots, k \quad **$$

Consider $A(k+1)$, $k \ge 2$: $\quad u_{k+1} = 6u_k - 8u_{k-1}$

$$\begin{aligned}\therefore u_{k+1} &= 6(4^k - 2^k) - 8(4^{k-1} - 2^{k-1}) \qquad \textit{if } A(n) \textit{ is true for } n \le k, \textit{ using } ** \\ &= 6.4^k - 6.2^k - 2.4.4^{k-1} + 4.2.2^{k-1} \\ &= (6-2)4^k - (6-4)2^k \\ &= 4^{k+1} - 2^{k+1}\end{aligned}$$

Hence if $A(n)$ is true for $n \le k$ (k some integer, $k \ge 2$) , then $A(k+1)$ is true. But $A(1)$ and $A(2)$ are true, and hence $A(3)$ is true; then $A(n)$ is true for $n = 1, 2, 3$ and hence $A(4)$ is true and so on. Hence by mathematical induction, $A(n)$ is true for all positive integers $n \ge 1$.

(ii) $S_n = \sum_{k=1}^{n} u_k = \sum_{k=1}^{n} (4^k - 2^k) = \sum_{k=1}^{n} 4^k - \sum_{k=1}^{n} 2^k$

$\sum_{k=1}^{n} 4^k = \dfrac{4(4^n - 1)}{4-1} = \frac{4}{3}(4^n - 1)$ (sum of n terms of geometric progression, $a = 4$, $r = 4$)

$\sum_{k=1}^{n} 2^k = \dfrac{2(2^n - 1)}{2-1} = 2(2^n - 1)$ (sum of n terms of geometric progression, $a = 2$, $r = 2$)

$\therefore S_n = \frac{4}{3}(4^n - 1) - 2(2^n - 1) = \frac{1}{3}\, 2^{2n+2} - \frac{4}{3} - 2^{n+1} + 2 = \frac{1}{3}\, 2^{2n+2} - 2^{n+1} + \frac{2}{3}$

Question 8

(a) **Outcomes Assessed:** (i) **H5** (ii) **PE3, E2, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for differentiation • one mark for simplification to obtain required result	2
(ii) • one mark for using $\frac{dy}{dx} < 0$ to deduce function is decreasing for $0 < x < \frac{\pi}{2}$ • one mark for establishing $y = 0$ when $x = 0$ • one mark for deducing the required inequality	3

Answer

(i) $y = x - \ln(\sec x + \tan x)$, $\quad 0 \le x < \frac{\pi}{2}$

$$\begin{aligned}\frac{dy}{dx} &= 1 - \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= 1 - \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \\ &= 1 - \sec x\end{aligned}$$

(ii) $x = 0 \Rightarrow y = 0 - \ln(1+0) = 0$

$\frac{dy}{dx} = 0$ for $x = 0$, and $\frac{dy}{dx} < 0$ for $0 < x < \frac{\pi}{2}$

Hence $y = x - \ln(\sec x + \tan x)$ is a decreasing function, and hence $y < 0$, for $0 < x < \frac{\pi}{2}$.

$x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$.

(b) **Outcomes Assessed:** (i) **H5** (ii) **H5, E2** (iii) **H5**

Marking Guidelines

Criteria	Marks
(i) • one mark for establishing required identity	1
(ii) • one mark for repeated use of this identity • one mark for simplification to obtain stated result	2
(iii) • one mark for using this result to rearrange integrand • one mark for evaluation of integral	2

Answer

(i) $\left.\begin{array}{l}\sin(A+B)=\sin A\cos B+\cos A\sin B\\ \sin(A-B)=\sin A\cos B-\cos A\sin B\end{array}\right\} \Rightarrow \begin{array}{l}\sin(A+B)-\sin(A-B)=2\cos A\sin B\\ \dfrac{\sin(A+B)-\sin(A-B)}{2\sin B}=\cos A\end{array}$

(ii) Let $A=(2n-1)x$, $B=x$. Then

$$\left.\begin{array}{l}A=(2n-1)x\\ B=x\end{array}\right\} \Rightarrow \cos(2n-1)x=\frac{\sin 2nx-\sin 2(n-1)x}{2\sin x}=\frac{\sin 2nx}{2\sin x}-\frac{\sin 2(n-1)x}{2\sin x}$$

Hence

$$\cos x+\cos 3x+\cos 5x+\ldots+\cos(2n-3)x+\cos(2n-1)x$$

$$=\left(\frac{\sin 2x}{2\sin x}-\frac{\sin 0}{2\sin x}\right)+\left(\frac{\sin 4x}{2\sin x}-\frac{\sin 2x}{2\sin x}\right)+\left(\frac{\sin 6x}{2\sin x}-\frac{\sin 4x}{2\sin x}\right)+\ldots$$

$$\ldots+\left(\frac{\sin 2(n-1)x}{2\sin x}-\frac{\sin 2(n-2)x}{2\sin x}\right)+\left(\frac{\sin 2nx}{2\sin x}-\frac{\sin 2(n-1)x}{2\sin x}\right)$$

$$\therefore\ \cos x+\cos 3x+\ldots+\cos(2n-1)x=\frac{\sin 2nx}{2\sin x}$$

(iii)

$$\int_0^{\frac{\pi}{2}}\frac{\sin 8x}{\sin x}\,dx=2\int_0^{\frac{\pi}{2}}(\cos x+\cos 3x+\cos 5x+\cos 7x)\,dx=2\left[\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\tfrac{1}{7}\sin 7x\right]_0^{\frac{\pi}{2}}$$

$$\therefore\ \int_0^{\frac{\pi}{2}}\frac{\sin 8x}{\sin x}\,dx=2\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}\right)=\frac{152}{105}$$

(c) **Outcomes Assessed:** (i) **PE3, E2** (ii) **E2, E9**

Marking Guidelines

Criteria	Marks
(i) • one mark for obtaining equations for A and B • one mark for values of A and B	2
(ii) • one mark for expressing $2^{14}+1$ in form $4\times 8^4+1$ • one mark for using the polynomial factorisation to obtain factors 145×113 • one mark for prime factors 5, 29, 113	3

Answer

(i) $4x^4+1\equiv(2x^2+Ax+1)(2x^2+Bx+1)\equiv 4x^4+2(A+B)x^3+(AB+4)x^2+(A+B)x+1$

Equating coefficients: $\left.\begin{array}{l}A+B=0\\ AB+4=0\end{array}\right\} \Rightarrow \begin{array}{l}B=-A\\ -A^2+4=0\end{array}\quad \therefore\ \begin{array}{c}A=2,\ B=-2\\ \text{or}\\ A=-2,\ B=2\end{array}$

Hence $4x^4+1\equiv(2x^2+2x+1)(2x^2-2x+1)$ **

(ii) $2^{14}+1=4(2^3)^4+1=\left\{2(2^3)^2+2(2^3)+1\right\}\left\{2(2^3)^2-2(2^3)+1\right\}$, putting $x=(2^3)$ in **.

$$\therefore\ 2^{14}+1=(2\times 64+16+1)(2\times 64-16+1)=145\times 113=5\times 29\times 113$$