Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2001
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Friday 17 August 2001

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided separately
- All necessary working should be shown in every question

Total marks **(120)**

- Attempt Questions 1 – 8
- All questions are of equal value

EXAMINERS

Graham Arnold	Terra Sancta College, Nirimba
Denise Arnold	Patrician Brothers' College, Blacktown

Question 1 **Begin a new page** **Marks**

(a) $P(x)=(x+2)(x-1)(x-3)$

(i) Sketch $y=P(x)$ showing the intercepts on the coordinate axes. **1**

(ii) On separate diagrams, sketch the graphs of $y=|P(x)|$, $y=P(|x|)$, $y=\dfrac{1}{P(x)}$ showing the intercepts on the coordinate axes and the equations of any asymptotes. **4**

(b) (i) $P(x_1, y_1)$ is a point on the curve $y=e^{-x}$. The tangent to the curve at P passes through the origin. Find the coordinates of P. **3**

(ii) Find the set of values of the real number k such that the equation $e^{-x}=kx$ has two real and distinct solutions. **2**

(c) Consider the function $f(x)=\ln(1+\cos x)$, $-2\pi \le x \le 2\pi$, where $x \neq \pi$, $x \neq -\pi$.

(i) Show that the function f is even and the curve $y=f(x)$ is concave down for all values of x in its domain. **3**

(ii) Sketch the graph of the curve $y=f(x)$. **2**

Question 2 **Begin a new page**

(a) Find all the complex numbers $z=a+ib$, a, b real, such that $|z|^2 - iz = 16-2i$. **3**

(b) (i) Find $\displaystyle\int \frac{e^x+1}{e^x}\,dx$. **1**

(ii) Find $\displaystyle\int \frac{x^2+x+1}{x(x^2+1)}\,dx$. **2**

(c) (i) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}\,dx$. **2**

(ii) Hence use the substitution $u=\frac{\pi}{2}-x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}\,dx$. **2**

(d) (i) If $I_n=\displaystyle\int_0^1 (1+x^2)^n\,dx$, $n=0, 1, 2, \ldots$ show that $(2n+1)I_n = 2^n + 2n\,I_{n-1}$ for $n=1, 2, 3, \ldots$ **3**

(ii) Hence find a reduction formula for $J_m=\displaystyle\int_0^{\frac{\pi}{4}} \sec^{2m}x\,dx$ **2**

Marks

Question 3 **Begin a new page**

(a) In an Argand Diagram, the point P representing the complex number z moves so that $|z-(1+i)|=1$.

(i) Sketch the locus of P. **1**

(ii) Shade the region where $|z-(1+i)|\leq 1$ and $0<\arg(z-i)<\frac{\pi}{4}$ **1**

(b) In an Argand Diagram, a regular hexagon $ABCDEF$, with the vertices taken in anticlockwise order, has its centre at the origin O and vertex A at $z=2$.

(i) Find the set of values of $\text{Im}(z)$ for points z on the hexagon. **1**

(ii) Find the set of values of $|z|$ for points z on the hexagon. **1**

(iii) If the hexagon is rotated in a clockwise direction about the origin through an angle of $45°$, find the value in modulus / argument form of the complex number which is represented by the new position of the vertex C. **1**

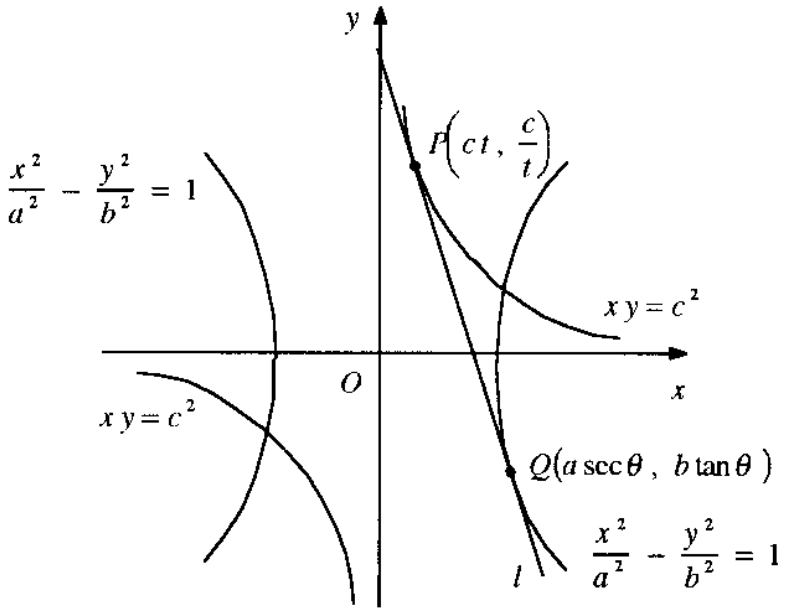
(c) (i) If $z=\cos\theta + i\sin\theta$, show that for positive integers n, $z^n+\frac{1}{z^n}=2\cos n\theta$ and $z^n-\frac{1}{z^n}=2i\sin n\theta$. Hence expand $\left(z+\frac{1}{z}\right)^4+\left(z-\frac{1}{z}\right)^4$ to show that $\cos^4\theta+\sin^4\theta=\frac{1}{4}(\cos 4\theta+3)$. **3**

(ii) By letting $x=\cos\theta$, show that the equation $8x^4+8\left(1-x^2\right)^2=7$ has roots $\pm\cos\frac{\pi}{12}$, $\pm\cos\frac{5\pi}{12}$. **2**

(iii) Deduce that $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$ have a product of $\frac{1}{4}$ and a sum of $\sqrt{\frac{3}{2}}$. **3**

(iv) Hence or otherwise find a surd expression for $\cos\frac{\pi}{12}$. **2**

Question 4 **Begin a new page** **Marks**



The line l is a common tangent to the hyperbolas $xy = c^2$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with points of contact P and Q respectively.

(i) Considering l as a tangent to $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$, show l has equation $x + t^2y = 2ct$. **2**

(ii) Considering l as a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $Q(a\sec\theta, b\tan\theta)$, show l has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **2**

(iii) Deduce that $\frac{\sec\theta}{a} = \frac{-\tan\theta}{bt^2} = \frac{1}{2ct}$. **1**

(iv) Write the coordinates of Q in terms of t, a, b and c, and show that $b^2t^4 + 4c^2t^2 - a^2 = 0$. **3**
Deduce that there are exactly two such common tangents to the hyperbolas.

(v) Copy the diagram and use the symmetry in the graphs to draw in the second common tangent with points of contact R on $xy = c^2$ and S on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **2**
Write the coordinates of R and S in terms of t, a, b and c.

(vi) Show that if $PQRS$ is a rhombus, then $b^2 = a^2$ and deduce that $t^2 < 1$. **2**

(vii) Show that if $PQRS$ is a square, then $t^4 + 2t^2 - 1 = 0$ and deduce that $2c^2 = a^2$. **3**
What is the relationship between the two hyperbolas if $PQRS$ is a square ?

Question 5 **Begin a new page** **Marks**

(a) $I=\int_0^{\pi} xe^x\cos x\ dx$ and $J=\int_0^{\pi} e^x\cos x\ dx$

(i) Use integration by parts to show that $I-J=-\int_0^{\pi} xe^x\sin x\ dx$. **2**

(ii) Differentiate xe^x and hence find $\int (x+1)e^x\ dx$. Hence or otherwise show that **2**

$$I+J=-\pi\ e^{\pi}+\int_0^{\pi} xe^x\sin x\ dx\ .$$

(iii) Evaluate I. **1**

(b) (i) On the same diagram and without using calculus, sketch the graphs of $y=e^{-x}$, $y=-e^{-x}$ and $y=e^{-x}\cos x$, $0\le x\le 2\pi$. Shade the region bounded by $y=e^{-x}$, $y=e^{-x}\cos x$ and $x=\pi$ for $x\ge 0$. **3**

(ii) The region shaded in (i) is rotated through one revolution about the line $x=\pi$. Use the method of cylindrical shells to show that the volume of the solid of revolution is given by **2**

$$V=2\pi\int_0^{\pi}(\pi-x)e^{-x}(1-\cos x)\ dx\ .$$

(iii) Use the substitution $u=\pi-x$ to show $V=2\pi e^{-\pi}\left\{\int_0^{\pi} ue^u\ du\ +I\right\}$, where I is as defined in (a). **2**

(iv) Hence find the volume of the solid. **3**

Question 6 **Begin a new page**

An object of mass m kg is dropped from rest from the top of a cliff 40 m above the water. Before the object reaches the water, the resistance to its motion has magnitude $\frac{1}{10}mv$ when the object has speed v ms^{-1}. After the object enters the water, the resistance to its motion has magnitude $\frac{1}{10}mv^2$. Take $g=10$ ms^{-2}.

(a) (i) Write an expression for $\ddot{x}$ before the object enters the water, where x metres is the distance the object has fallen in t seconds. **1**

(ii) Show $10\frac{dv}{dx}=\frac{100-v}{v}$, and show that the speed of the object as it enters the water is V ms^{-1} where V satisfies $\frac{V}{100}+\ln\left(1-\frac{V}{100}\right)+0{\cdot}04=0$. **3**

(iii) Show this equation has a solution for V between 20 and 30, and taking 25 as a first approximation, use Newton's Method to show that $V\approx 25{\cdot}7$ to one decimal place. **3**

Question 6 continued **Marks**

(b) (i) Write an expression for $\ddot{x}$ after the object enters the water. Deduce the object slows on entry to the water, and find its terminal velocity in the water. **3**

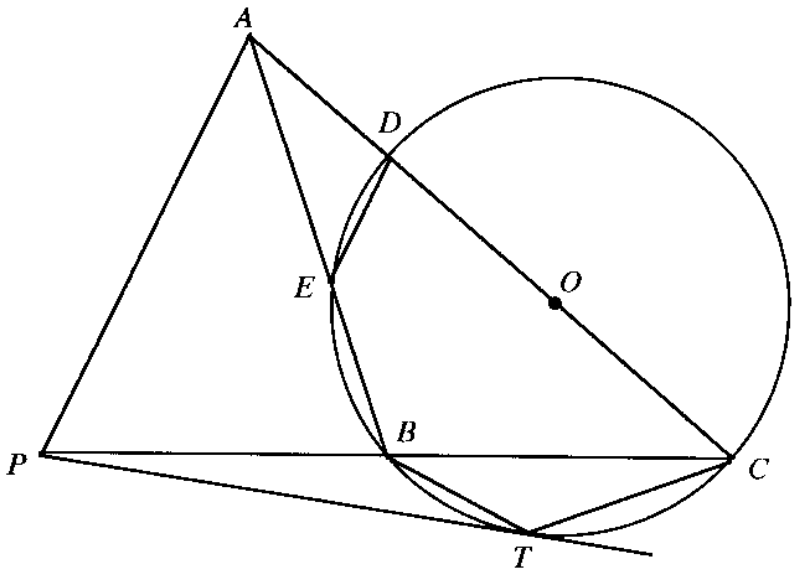
(ii) Show that t seconds after entering the water $10\frac{dv}{dt} = 100 - v^2$, and the velocity **3**

v ms^{-1} of the object is given by $2t = \ln\left\{\frac{(v+10)(V-10)}{(v-10)(V+10)}\right\}$, where V is the velocity on entry to the water calculated in (a).

(iii) How long after it enters the water will the body slow to 105% of its terminal velocity? **2**

Question 7 **Begin a new page**

(a)



A is a point outside a circle with centre O. P is a second point outside the circle such that $PT = PA$ where PT is a tangent to the circle at T. AO cuts the circle at D and C. PC cuts the circle at B. AB cuts the circle at E.

(i) Copy the diagram.

(ii) Show that $\Delta PBT \;|||\; \Delta PTC$. **2**

(iii) Show that $\Delta APB \;|||\; \Delta CPA$. **3**

(iv) Hence show that DE is parallel to AP. **3**

(b) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_1 = 2$, $u_2 = 12$ and $u_n = 6u_{n-1} - 8u_{n-2}$ for $n \geq 3$.

(i) Use Mathematical Induction to show that $u_n = 4^n - 2^n$ for $n \geq 1$. **4**

(ii) If $S_n = u_1 + u_2 + u_3 + \ldots + u_n$, find an expression for S_n in the form $S_n = a\,2^{2n+2} + b\,2^{n+1} + c$ where a, b, c are numerical constants. **3**

Marks

Question 8 **Begin a new page**

(a) (i) Given that $y = x - \ln(\sec x + \tan x)$, $0 \leq x < \frac{\pi}{2}$, show that $\frac{dy}{dx} = 1 - \sec x$. **2**

(ii) Hence show that $x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$. **3**

(b) (i) Show that $\frac{\sin(A+B) - \sin(A-B)}{2\sin B} = \cos A$. **1**

(ii) Hence show that **2**

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-3)x + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}$$

(iii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\, dx$. **2**

(c) (i) Find the values of the constants A and B such that **2**

$$4x^4 + 1 = (2x^2 + Ax + 1)(2x^2 + Bx + 1)$$

(ii) Hence find the prime factors of the integer $2^{14} + 1$. **3**