Mathematics Extension I CSSA HSC Trial Examination 2001
Marking Guidelines

Question 1

(a) **Outcomes Assessed: H5, H9**

Marking Guidelines

Criteria	Marks
• one mark for simplification of sum • one mark for value of sum	2

Answer:

$$\sum_{k=1}^{4}(-1)^k k! = -1! + 2! - 3! + 4! = 19$$

(b) **Outcomes Assessed: P4**

Marking Guidelines

Criteria	Marks
• one mark for values of gradients • one mark for value of $\tan\theta$ • one mark for size of angle	3

Answer:

$$\left.\begin{array}{r} AB \text{ has gradient } m_1 = 3 \\ x+2y+1=0 \text{ has gradient } m_2 = -\tfrac{1}{2} \end{array}\right\} \Rightarrow \tan\theta = \left|\frac{3-\left(-\frac{1}{2}\right)}{1+3\left(-\frac{1}{2}\right)}\right| = 7 \qquad \therefore \theta = 81^\circ 52'$$

(c) **Outcomes Assessed: (i) P5 (ii) PE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for showing $P(x)$ is odd (ii) • one mark for showing remainder is $-P(2)$ • one mark for value of remainder	3

Answer:

(i)

$$\begin{aligned} P(-x) &= (-x)^5 + a(-x)^3 + b(-x) \\ &= -x^5 - a x^3 - b x \\ &= -\left(x^5 + a x^3 + b x\right) \\ &= -P(x) \quad \text{for all } x \end{aligned}$$

$\therefore P(x)$ is odd.

(ii) When $P(x)$ is divided by $(x+2)$, remainder is $P(-2) = -P(2)$ since $P(x)$ is odd

$= -5$ since $P(2) = 5$

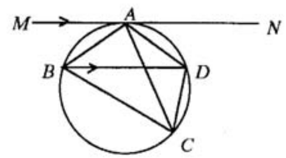
(d) **Outcomes Assessed: (i) (ii) PE3 (iii) PE3 (iv) H5, PE2, PE3**

Marking Guidelines

Criteria	Marks
(i) • no marks for copying diagram (ii) • one mark for reason (iii) • one mark for reason (iv) • one mark for showing $\angle MAB = \angle ABD$ • one mark for showing $\angle ACB = \angle ACD$	4

Answer:

(i)



(ii) $\angle ACB = \angle MAB$ because the angle between the tangent MA and the chord AB through the point of contact A is equal to the angle ACB in the alternate segment.

(iii) $\angle ACD = \angle ABD$ because the angles subtended in the same segment at B and C by the arc AD are equal.

(iv)

$\angle MAB = \angle ABD$ (equal alternate angles, $MN \parallel BD$)

$\angle ACB = \angle ACD$ ($\angle MAB = \angle ACB$, $\angle ABD = \angle ACD$)

$\therefore AC$ bisects $\angle BCD$

Question 2

(a) **Outcomes Assessed: P7, PE5**

Marking Guidelines

Criteria	Marks
• one mark for first derivative • one mark for second derivative using product rule.	2

Answer:

$$\frac{d}{dx}e^{x^2} = 2x e^{x^2} \qquad \frac{d^2}{dx^2}e^{x^2} = \frac{d}{dx}2x e^{x^2} = 2\left(e^{x^2}\right) + (2x)\left(2x e^{x^2}\right) = 2\left(1+2x^2\right)e^{x^2}$$

(b) **Outcomes Assessed: P4**

Marking Guidelines

Criteria	Marks
• one mark for equation in x • one mark for equation in y • one mark for coordinates of B	3

Answer:

$$\frac{5x - 3\times(-1)}{5-3} = 14 \Rightarrow 5x + 3 = 28 \qquad \therefore x = 5$$

$$\frac{5y - 3\times(4)}{5-3} = -6 \Rightarrow 5y - 12 = -12 \qquad \therefore y = 0$$

$\therefore B(5, 0)$

(c) **Outcomes Assessed: PE3**

Marking Guidelines

Criteria	Marks
• one mark for number of arrangements of vowels • one mark for number of arrangements of consonants • one mark for total number of arrangements	3

Answer:

The vowels (E, E, I, O) can be arranged in positions 2, 4, 6, 8 in $\frac{4!}{2!} = 12$ ways.

The consonants (N,N, S, T, X) can be arranged in positions 1, 3, 5, 7, 9 in $\frac{5!}{2!} = 60$ ways.

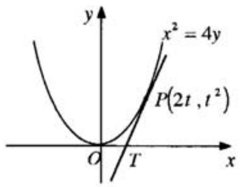
Hence the total number of arrangements is $12 \times 60 = 720$.

(d) **Outcomes Assessed:** (i) **PE3, PE4** (ii) **PE3** (iii) **PE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for equation of tangent (ii) • one mark for coordinates of T • one mark for coordinates of M (iii) • one mark for equation of locus	4

Answer:



(i) $y = \frac{1}{4}x^2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}x$

$\therefore$ tangent at $P(2t, t^2)$ has gradient $\frac{1}{2}(2t) = t$

and equation $y - t^2 = t(x - 2t)$

$tx - y - t^2 = 0$

(ii) At T, $y = 0 \Rightarrow tx - 0 - t^2 = 0 \Rightarrow x = t$

Hence T has coordinates $(t, 0)$, and

M is the midpoint of $P(2t, t^2)$ and $T(t, 0)$,

with coordinates $\left(\frac{2t+t}{2}, \frac{t^2+0}{2}\right) \equiv \left(\frac{3t}{2}, \frac{t^2}{2}\right)$.

(iii) At M, $x = \frac{3t}{2} \Rightarrow t = \frac{2x}{3}$

$\therefore y = \frac{1}{2}t^2 = \frac{1}{2}\left(\frac{2x}{3}\right)^2 = \frac{2x^2}{9}$

Hence the locus has equation $2x^2 = 9y$.

Question 3

(a) **Outcomes Assessed:** (i) **P4** (ii) **PE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for expansion and expressions for $\cos 2A$, $\sin 2A$ • one mark for simplification to obtain final expression for $\cos 3A$ in terms of $\cos A$ (ii) • one mark for expressing $2\cos 3A$ in terms of $\left(x + \frac{1}{x}\right)$ • one mark for binomial expansion of $\left(x + \frac{1}{x}\right)^3$ • one mark for simplification to obtain final expression for $\cos 3A$ in terms of x	5

Answer:

(i)

$\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 4\cos^3 A - 3\cos A$

(ii)

$2\cos 3A = 8\cos^3 A - 6\cos A$

$= (2\cos A)^3 - 3(2\cos A)$

$= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$

$= x^3 + 3x + \frac{3}{x} + \left(\frac{1}{x}\right)^3 - 3x - \frac{3}{x}$

$= x^3 + \frac{1}{x^3}$

(b) **Outcomes Assessed:** (i) **P5, HE4** (ii) **P5, HE4** (iii) **P4**

Marking Guidelines

Criteria	Marks
(i) • one mark for finding the inverse function • one mark for the domain of the inverse function (ii) • one mark for the graph of $y = f(x)$ and intercepts • one mark for the graph of $y = f^{-1}(x)$ and intercepts • one mark for the line $y = x$ passing through the point of intersection (iii) • one mark for the equation • one mark for the coordinates of the point of intersection	7

Answer:

(i)

$y = \sqrt{x+6}$

$y^2 = x + 6$

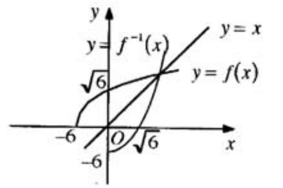
$x = y^2 - 6$

Interchanging x and y gives $y = x^2 - 6$

$\therefore f^{-1}(x) = x^2 - 6$

Range of $f(x)$ is $\{y : y \geq 0\}$ $\Rightarrow$ Domain of $f^{-1}(x)$ is $\{x : x \geq 0\}$

(ii)



(iii) Where $y = f(x)$, $y = f^{-1}(x)$, $y = x$ intersect,

$f^{-1}(x) = x \Rightarrow x^2 - 6 = x$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

But $x \neq -2$ (outside domain). $\therefore x = 3$

Hence intersection point of the curves is $(3, 3)$.

Question 4

(a) **Outcomes Assessed: HE2**

Marking Guidelines

Criteria	Marks
• one mark for establishing the truth of $S(1)$ • one mark for $S(k)$ true $\Rightarrow 5^k + 2(11^k) = 3M$ for some integer M. • one mark for $5^{k+1} + 2(11^{k+1}) = 5(5^k) + 22(11^k)$ • one mark for deducing $S(k)$ true $\Rightarrow S(k+1)$ true • one mark for deducing $S(n)$ true for all integers $n \ge 1$	5

Answer:

Define the sequence of statements $S(n)$: $5^n + 2(11^n)$ *is a multiple of* 3, $n = 1, 2, 3, \ldots$

Consider $S(1)$: $5^1 + 2(11^1) = 27 = 3\times 9 \quad \therefore\ S(1)$ is true.

If $S(k)$ *is true*, then $5^k + 2(11^k) = 3M$ for some integer M. **

Consider $S(k+1)$: $5^{k+1} + 2(11^{k+1}) = 5(5^k) + 22(11^k) = 5\{(5^k) + 2(11^k)\} + 12(11^k)$

$\therefore\ 5^{k+1} + 2(11^{k+1}) = 5(3M) + 12(11^k) = 3\{5M + 4(11^k)\}$ *if* $S(k)$ *is true, using* **

But M and k integral $\Rightarrow \{5M + 4(11^k)\}$ is an integer.

$\therefore\ S(k)$ true $\Rightarrow S(k+1)$ true, $k = 1, 2, 3, \ldots$

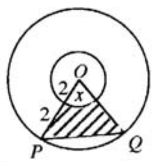
Hence $S(1)$ is true, and if $S(k)$ is true, then $S(k+1)$ is true. $\therefore\ S(2)$ is true, and then $S(3)$ is true, and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers n.

(b) **Outcomes Assessed**: (i) **H5** (ii) **P5, H2** (iii) **PE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for areas of small circle sector and triangle OPQ • one mark for equating expression for shaded area to $\frac{1}{16}$ of large circle area • one mark for simplification to find equation in required form (ii) • one mark for showing $f(0\cdot5), f(0\cdot6)$ have opposite signs • one mark for using continuity of $f(x)$ to deduce $0\cdot5 < \alpha < 0\cdot6$ (iii) • one mark for expression for second approximation • one mark for calculation of second approximation	7

Answer:



(i)

Area of $\Delta\, POQ = \frac{1}{2}(4^2)\sin x$

Area small circle sector $= \frac{1}{2}(2^2)\, x$

$\therefore$ shaded area $= 8\sin x - 2x$

$\therefore\ 8\sin x - 2x = \frac{1}{16}\pi(4^2) = \pi$

$8\sin x - 2x - \pi = 0$

(ii) Let $f(x) = 8\sin x - 2x - \pi$. Then $f(0\cdot5) \approx -0\cdot31 < 0$ and $f(0\cdot6) \approx 0\cdot18 > 0$. Hence, since $f(x)$ is continuous, $f(\alpha) = 0$ for some $0\cdot5 < \alpha < 0\cdot6$.

(iii) Taking a first approximation $\alpha \approx 0\cdot6$, Newton's method gives a second approximation

$$\alpha \approx 0\cdot6 - \frac{f(0\cdot6)}{f'(0\cdot6)}$$

$$= 0\cdot6 - \frac{8\sin(0\cdot6) - 2(0\cdot6) - \pi}{8\cos(0\cdot6) - 2}$$

$\approx 0\cdot56$ to 2 decimal places.

Question 5

(a) **Outcomes Assessed: HE6**

Marking Guidelines

Criteria	Marks
• one mark for change of limits • one mark for change of variable • one mark for integration • one mark for evaluation	4

Answer:

Let $I = \int_1^{49} \frac{1}{4(x+\sqrt{x})}\, dx$

$u^2 = x$, $u > 0$

$2u = \frac{dx}{du} \Rightarrow dx = 2u\, du$

$x = 1 \Rightarrow u = 1, \quad x = 49 \Rightarrow u = 7$

Then $I = \int_1^7 \frac{1}{4(u^2+u)}\, 2u\, du$

$= \int_1^7 \frac{1}{2(u+1)}\, du$

$= \frac{1}{2}\left[\ln(u+1)\right]_1^7$

$\therefore\ I = \frac{1}{2}(\ln 8 - \ln 2) = \frac{1}{2}\ln 4 = \ln 2$

(b) **Outcomes Assessed: H5**

Marking Guidelines

Criteria	Marks
• one mark for expressing $\sin^2 x$ in terms of $\cos 2x$ • one mark for integration, including constant of integration • one mark for evaluation of $f\left(\frac{\pi}{4}\right), f\left(\frac{3\pi}{4}\right)$ • one mark for value of difference	4

Answer:

$\frac{dy}{dx} = \sin^2 x$

$= \frac{1}{2}(1 - \cos 2x)$

$y = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$, c *constant*

$f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

$\therefore\ f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)$

$= \left(\frac{3\pi}{8} - \frac{1}{4}\sin\frac{3\pi}{2} + c\right) - \left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + c\right)$

$= \left(\frac{3\pi}{8} + \frac{1}{4} + c\right) - \left(\frac{\pi}{8} - \frac{1}{4} + c\right)$

$= \frac{\pi}{4} + \frac{1}{2}$

(c) **Outcomes Assessed**: (i) **HE3** (ii) **H5, HE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for finding the period of the motion (ii) • one mark for expressing v^2 in terms of t • one mark for expressing v^2 in terms of x • one mark for the value of the speed.	4

Answer:

(i) Period is $2\pi \div \frac{\pi}{2} = 4$ seconds

(ii)

$x = 5\cos\frac{\pi}{2}t$

$v = \frac{dx}{dt} = 5\left(-\frac{\pi}{2}\sin\frac{\pi}{2}t\right)$

$v^2 = \left(\frac{\pi^2}{4}\right).25\sin^2\frac{\pi}{2}t$

$v^2 = \left(\frac{\pi^2}{4}\right).25\left(1 - \cos^2\frac{\pi}{2}t\right)$

$= \frac{\pi^2}{4}\left(25 - 25\cos^2\frac{\pi}{2}t\right)$

$v^2 = \frac{\pi^2}{4}\left(25 - x^2\right)$

$x = 4 \Rightarrow v^2 = \frac{\pi^2}{4}(25 - 16) = \frac{9\pi^2}{4}$

Speed is $\frac{3\pi}{2}$ ms^{-1}

Question 6

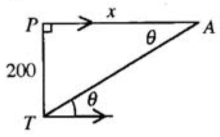
(a) **Outcomes Assessed**: (i) **P4, HE4** (ii) **HE4, HE5** (iii) **H5**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for θ (ii) • one mark for expression for $\frac{d\theta}{dx}$ • one mark for expression for $\frac{d\theta}{dt}$ (iii) • one mark for value of $\frac{d\theta}{dt}$ • one mark for value of θ	5

Answer:

(i)



$\angle TAP = \theta$

(alt. $\angle$ s, parallel lines)

$\tan\theta = \frac{200}{x}$

$\theta = \tan^{-1}\frac{200}{x}$

(ii)

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{200}{x}\right)^2}\left(-\frac{200}{x^2}\right) = \frac{-200}{x^2+40000}$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{-200}{x^2+40000}\ (-80)$$

$$\therefore \frac{d\theta}{dt} = \frac{16\,000}{x^2+40\,000}$$

(iii) When $\theta = \frac{\pi}{4}$, $TP = AP \Rightarrow x = 200$, and $\frac{d\theta}{dt} = \frac{16000}{(200)^2+40000} = 0\cdot2$ radians per second.

Hence θ is increasing at $11°\ s^{-1}$ (correct to the nearest degree)

(b) **Outcomes Assessed**: (i) **HE5** (ii) **H3, H5, HE4** (iii) **HE3, HE7**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for a in terms of x (ii) • one mark for expressing t as an integral with respect to x • one mark for integration to find t in terms of x • one mark for expression for x^2 in terms of t (iii) • one mark for graph of x^2 as a function of t • one mark for limiting values of x, v, a • one mark for description of limiting behaviour in words	7

Answer:

(i)

$$v^2 = \left(\frac{32}{x} - \frac{x}{2}\right)^2 = \frac{1024}{x^2} - 32 + \frac{x^2}{4}$$

$$a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{1024}{x^2} - 32 + \frac{x^2}{4}\right)$$

$$\therefore\ a = \frac{-1024}{x^3} + \frac{x}{4}$$

(ii)

$$\frac{dx}{dt} = v = \frac{32}{x} - \frac{x}{2} = \frac{64-x^2}{2x}$$

$$\therefore \frac{dt}{dx} = \frac{2x}{64-x^2}$$

$$t = \int \frac{2x}{64-x^2}\,dx$$

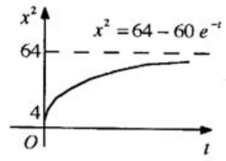
(ii) Cont.

$$t = -\ln\left(64-x^2\right) + c\ ,\quad \left.\begin{matrix} t=0 \\ x=2 \end{matrix}\right\} \Rightarrow c = \ln 60$$

$$-t = \ln\left(\frac{64-x^2}{60}\right)\ ,\qquad e^{-t} = \frac{64-x^2}{60}$$

$$\therefore x^2 = 64 - 60e^{-t}$$

(iii)



As $t \to \infty$, $x \to 8^-$, $v \to \frac{32}{8} - \frac{8}{2} = 0^+$, $a \to \frac{-1024}{512} + \frac{8}{4} = 0^-$

Hence the particle is moving right and slowing down as it approaches its limiting position 8 metres to the right of O.

Question 7

(a) **Outcomes Assessed**: (i) **HE3** (ii) **HE3**

Marking Guidelines

Criteria	Marks
(i) • one mark for value of probability (ii) • one mark for expression for probability of two 6's on first roll and no 6's on second • one mark for expression for probability of one 6 on first roll and one 6 on second • one mark for expression for probability of no 6's on first roll and two 6's on second • one mark for value of probability	5

Answer:

(i) $P(\textit{one 6 on first roll}) = {}^4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 \approx 0\cdot39$ (to 2 decimal places)

(ii) $P(\textit{two 6 s on first roll and no 6 s on second roll}) = {}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 \times {}^2C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^2 \approx 0\cdot0804$

$P(\textit{one 6 on first roll and one 6 on second roll}) = {}^4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 \times {}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 \approx 0\cdot1340$

$P(\textit{no 6 s on first roll and two 6 s on second roll}) = {}^4C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 \times {}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 \approx 0\cdot0558$

$\therefore\ P(\textit{two 6 s overall}) \approx 0\cdot0804 + 0\cdot1340 + 0\cdot0558 \approx 0\cdot27$ (to 2 decimal places)

(b) **Outcomes Assessed**: (i) **HE3** (ii) **HE3** (iii) **P4, H2** (iv) **P4, H2**

Marking Guidelines

Criteria	Marks
(i) • one mark for expressions for x and y in terms of θ and t (ii) • one mark for expression for y in terms of x • one mark for rearrangement as quadratic in $\tan\theta$ (iii) • one mark for discriminant in terms of X and Y • one mark for using discriminant > 0 to give required inequality (iv) • one mark for the values of the sum and product of $\tan\alpha$, $\tan\beta$ in terms of X • one mark for the value of $\alpha+\beta$	7

Answer:

(i) $x = 50\,t\cos\theta$ and $y = 50\,t\sin\theta - 5t^2$

(ii)

$$t = \frac{x}{50\cos\theta} \Rightarrow y = x\frac{\sin\theta}{\cos\theta} - \frac{5x^2}{2500\cos^2\theta}$$

$$500\,y = 500\,x\tan\theta - x^2\sec^2\theta$$

$$= 500\,x\tan\theta - x^2\left(1+\tan^2\theta\right)$$

$$= 500\,x\tan\theta - x^2 - x^2\tan^2\theta$$

$$\therefore\ x^2\tan^2\theta - 500\,x\tan\theta + \left(x^2 + 500\,y\right) = 0$$

(iii) Projectile passes through the point (X, Y) if $\tan\theta$ satisfies the quadratic equation

$$X^2\tan^2\theta - 500\,X\tan\theta + \left(X^2 + 500\,Y\right) = 0$$

This equation has two distinct solutions for $\tan\theta$, and hence for θ, provided its discriminant $\Delta > 0$.

$$\Delta = (-500\,X)^2 - 4\,X^2\left(X^2 + 500\,Y\right)$$

$$= 4X^2\left(62500 - X^2 - 500\,Y\right)$$

$$\therefore\ \Delta > 0 \quad \text{provided} \quad 500Y < 62500 - X^2$$

(iv) If the projectile passes through the point (X, X) where $500X < 62500 - X^2$, then the equation $X^2\tan^2\theta - 500\,X\tan\theta + \left(X^2 + 500\,X\right) = 0$ has two distinct real roots $\tan\alpha$, $\tan\beta$ where

$$\tan\alpha + \tan\beta = \frac{500\,X}{X^2} = \frac{500}{X} \quad \text{and} \quad \tan\alpha\,\tan\beta = \frac{X^2 + 500\,X}{X^2} = 1 + \frac{500}{X}$$

$$\therefore\ \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} = \frac{500}{X} \div \left(-\frac{500}{X}\right) = -1$$

Since $0 < \alpha + \beta < \pi$, $\alpha + \beta = \frac{3\pi}{4}$.