Catholic 3U 2001

Marks

Question 1 **Begin a new page**

(a) Find the value of $\sum_{k=1}^{4} (-1)^k k!$ 2

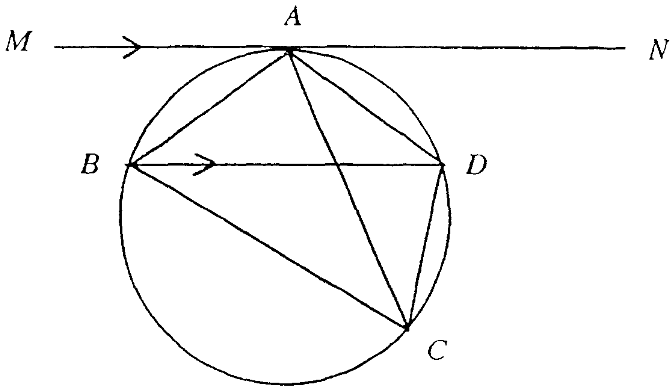
(b) $A(-2,-5)$ and $B(1,4)$ are two points. Find the acute angle θ between the line AB and the line $x+2y+1=0$, giving the answer correct to the nearest minute. 3

(c) The polynomial $P(x)=x^5+ax^3+bx$ leaves a remainder of 5 when it is divided by $(x-2)$, where a and b are numerical constants.

(i) Show that $P(x)$ is odd. 1

(ii) Hence find the remainder when $P(x)$ is divided by $(x+2)$. 2

(d)



$ABCD$ is a cyclic quadrilateral. The tangent at A to the circle through A, B, C and D is parallel to BD.

(i) Copy the diagram.

(ii) Give a reason why $\angle ACB = \angle MAB$. 1

(iii) Give a reason why $\angle ACD = \angle ABD$. 1

(iv) Hence show that AC bisects $\angle BCD$. 2

Marks

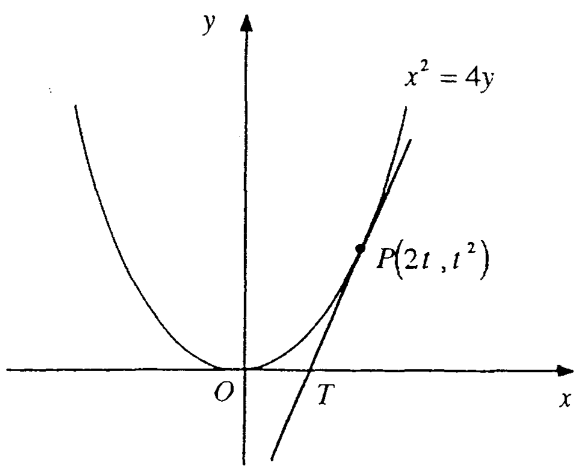
Question 2 **Begin a new page**

(a) Find $\dfrac{d^2}{dx^2}\, e^{x^2}$. 2

(b) $A(-1,4)$ and $B(x,y)$ are two points. The point $P(14,-6)$ divides the interval AB externally in the ratio $5:3$. Find the coordinates of B. 3

(c) Find the number of ways in which the letters of the word EXTENSION can be arranged in a straight line so that no two vowels are next to each other. 3

(d)



$P(2t, t^2)$ is a variable point which moves on the parabola $x^2 = 4y$. The tangent to the parabola at P cuts the x axis at T. M is the midpoint of PT.

(i) Show that the tangent PT has equation $tx - y - t^2 = 0$. 1

(ii) Show that M has coordinates $\left(\dfrac{3t}{2}, \dfrac{t^2}{2}\right)$. 2

(iii) Hence find the Cartesian equation of the locus of M as P moves on the parabola. 1

Marks

Question 3 **Begin a new page**

(a) (i) By expanding $\cos(2A + A)$, show that $\cos 3A = 4\cos^3 A - 3\cos A$. 2

(ii) Hence show that if $2\cos A = x + \frac{1}{x}$ then $2\cos 3A = x^3 + \frac{1}{x^3}$. 3

(b) The function $f(x)$ is given by $f(x) = \sqrt{x+6}$ for $x \geq -6$.

(i) Find the inverse function $f^{-1}(x)$ and find its domain. 2

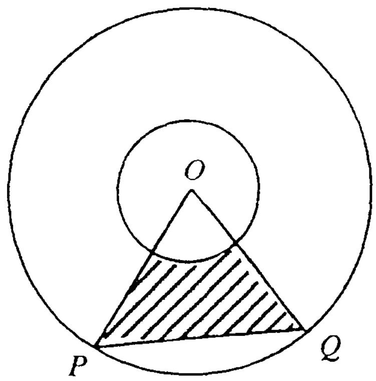
(ii) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the intercepts on the coordinate axes. Draw in the line $y = x$. 3

(iii) Show that the x coordinates of any points of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfy the equation $x^2 - x - 6 = 0$. Hence find any points of intersection of the two graphs. 2

Question 4 **Begin a new page**

(a) Use Mathematical Induction to show that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n. 5

(b)



Two concentric circles with centre O have radii 2 cm and 4 cm. The points P and Q lie on the larger circle and $\angle POQ = x$, where $0 < x < \frac{\pi}{2}$.

(i) If the area A cm^2 of the shaded region is $\frac{1}{16}$ the area of the larger circle, show that x satisfies the equation $8\sin x - 2x - \pi = 0$. 3

(ii) Show that this equation has a solution $x = \alpha$, where $0 \cdot 5 < \alpha < 0 \cdot 6$. 2

(iii) Taking $0 \cdot 6$ as a first approximation for α, use one application of Newton's Method to find a second approximation, giving the answer correct to two decimal places. 2

Marks

Question 5 **Begin a new page**

(a) Evaluate $\int_1^{49} \frac{1}{4\left(x+\sqrt{x}\right)}\,dx$ using the substitution $u^2 = x$, $u > 0$. Give the answer in simplest exact form. **4**

(b) At any point on the curve $y = f(x)$, the gradient function is given by $\frac{dy}{dx} = \sin^2 x$. **4**

Find the value of $f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right)$.

(c) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has velocity v metres per second, and displacement x metres from a fixed point O on the line, where $x = 5\cos\frac{\pi t}{2}$.

(i) Find the period of the motion. **1**

(ii) Find an expression for v in terms of t, and hence show that $v^2 = \frac{\pi^2}{4}\left(25 - x^2\right)$. **3**

Find the speed of the particle when it is 4 metres to the right of O.

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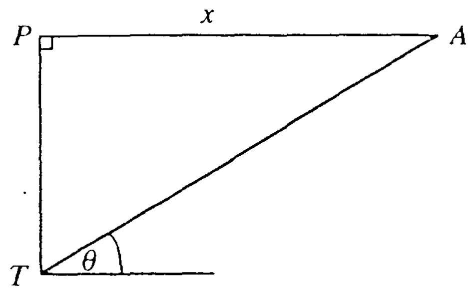
Question 6 **Begin a new page**

(a)



A person on horizontal ground is looking at an aeroplane A through a telescope T. The aeroplane is approaching at a speed of $80\ \text{ms}^{-1}$ at a constant altitude of 200 metres above the telescope. When the horizontal distance of the aeroplane from the telescope is x metres, the angle of elevation of the aeroplane is θ radians.

(i) Show that $\theta = \tan^{-1}\dfrac{200}{x}$. **1**

(ii) Show that $\dfrac{d\theta}{dt} = \dfrac{16000}{x^2+40000}$. **2**

(iii) Find the rate at which θ is changing when $\theta = \frac{\pi}{4}$, giving the answer in degrees per second correct to the nearest degree. **2**

(b) A particle moves in a straight line. At time t seconds its displacement is x metres from a fixed point O on the line, its acceleration is $a\ \text{ms}^{-2}$, and its velocity is $v\ \text{ms}^{-1}$ where v is given by $v = \dfrac{32}{x} - \dfrac{x}{2}$.

(i) Find an expression for a in terms of x. **1**

(ii) Show that $t = \displaystyle\int \frac{2x}{64-x^2}\,dx$, and hence show that $x^2 = 64 - 60\,e^{-t}$. **3**

(iii) Sketch the graph of x^2 against t and describe the limiting behaviour of the particle. **3**

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Question 7 **Begin a new page**

(a) Four fair dice are rolled. Any die showing 6 is left alone, while the remaining dice are rolled again.

(i) Find the probability (correct to 2 decimal places) that after the first roll of the dice, exactly one of the four dice is showing 6. **1**

(ii) Find the probability (correct to 2 decimal places) that after the second roll of the dice exactly two of the four dice are showing 6. **4**

b) A particle is projected from a point O with speed 50 ms^{-1} at an angle of elevation θ, and moves freely under gravity, where $g = 10 \text{ ms}^{-2}$.

(i) Write down expressions for the horizontal and vertical displacements of the particle at time t seconds referred to axes Ox and Oy. **1**

(ii) Hence show that the equation of the path of the projectile, given as a quadratic equation in $\tan\theta$, is $x^2 \tan^2\theta - 500x \tan\theta + \left(x^2 + 500y\right) = 0$. **2**

(iii) Hence show that there are two values of θ, $0 < \theta < \frac{\pi}{2}$, for which the projectile passes through a given point (X, Y) provided that $500\,Y < 62500 - X^2$. **2**

(iv) If the projectile passes through the point (X, X) whose coordinates satisfy this inequality, and the two values of θ are α and β, find expressions in terms of X for $\tan\alpha + \tan\beta$ and $\tan\alpha \tan\beta$, and hence show that $\alpha + \beta = \frac{3\pi}{4}$. **2**