BARKER COLLEGE 4U TRIAL 2001

Question 1

(a) $\int \frac{dx}{x^2-16x+80}$

$= \int \frac{dx}{(x-8)^2+16}$ ✓

$= \frac{1}{4}\tan^{-1}\left(\frac{x-8}{4}\right)+C$ ✓

(b) (i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$

$= \int_0^1 \frac{2}{(t+1)^2}dt$ ✓ let $\tan\frac{\theta}{2}=t$

$= -2\left[\frac{1}{t+1}\right]_0^1$ ✓ $\frac{dt}{d\theta}=\frac{1}{2}(1+t^2)$

$= 1$ ✓

(ii) $I = \int_0^2 \frac{8\,dx}{(x+2)(x^2+4)}$

$\frac{1}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$A=\frac{1}{8},\ B=-\frac{1}{8},\ C=\frac{1}{4}$ ✓

$I = 8\int_0^2 \frac{1}{8}\left(\frac{1}{x+2}\right) + \frac{-\frac{1}{8}x+\frac{1}{4}}{x^2+4}\,dx$

$= \left[\ln(x+2)\right]_0^2 - \frac{1}{2}\int_0^2 \frac{2x}{x^2+4}dx$ ✓

$+ 2\int \frac{1}{x^2+4}dx$

$= \ln 2 - \frac{1}{2}\left[\ln(x^2+4)\right]_0^2 + \left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$ ✓

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$ ✓

(iii) $I = \int_0^{\pi} e^x\cos x\,dx$

$= e^x\cos x + \int e^x\sin x\,dx$ ✓

$= e^x\cos x + \left[\sin x\, e^x - \int e^x\cos x\,dx\right]$

$= e^x\cos x + \sin x\, e^x - I$ ✓

$\therefore 2I = \left[e^x(\cos x+\sin x)\right]_0^{\pi}$

$I = \frac{1}{2}\left[-e^{\pi}-1\right]$ ✓

(c) $\int \frac{2x}{\sqrt{4x-x^2}}\,dx$ let $u=x-2$, $du=dx$

$= \int \frac{2(u+2)}{\sqrt{4(u+2)-(u+2)^2}}\,du$ ✓

$= 2\int \frac{u+2}{\sqrt{4-u^2}}\,du$

$= 2\int \frac{u}{\sqrt{4-u^2}}\,du + 4\int \frac{1}{\sqrt{4-u^2}}\,du$

$= -2\sqrt{4-u^2} + 4\sin^{-1}\left(\frac{u}{2}\right)$ ✓

$= -2\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)+C$ ✓

Question 2

(a) $(x+iy)^2 = -3+4i$

$x^2-y^2+2xyi = -3+4i$

$\left.\begin{array}{l} xy=2 \\ x^2-y^2=-3 \end{array}\right\}$ ✓

Solving simultan...

$\left.\begin{array}{l} x=2,\ y=1 \\ x=-1,\ y=-2 \end{array}\right\}$ ✓

(b) $z^2-3z+(3-i)=0$

$z = \frac{3\pm\sqrt{9-4(3-i)}}{2}$

$= \frac{3\pm\sqrt{-3+4i}}{2}$

$= 2+i$ or $1-i$ ✓

Question 2

(b) (i) $\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ ✓

$\sqrt{3}-i = 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$ ✓

(ii) $(\sqrt{3}+i)^{15}+(\sqrt{3}-i)^{15}$

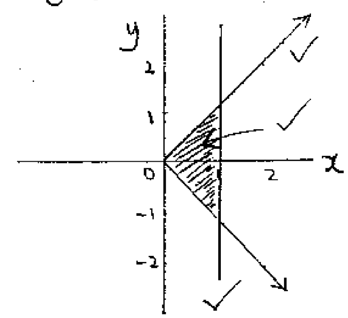
$= 2^{15}\left(\cos\frac{15\pi}{6}+i\sin\frac{15\pi}{6}\right)$

$+ 2^{15}\left(\cos\frac{15\pi}{6}-i\sin\frac{15\pi}{6}\right)$

$= 2\times 2^{15}\cos\frac{15\pi}{6}$

$= 0$ ✓

(c) (i)



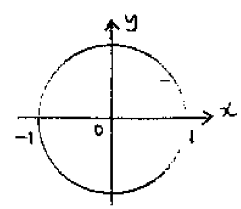
(ii) $Re\left(z-\frac{1}{z}\right)=0$

$Re\left(x+iy-\frac{x-iy}{x^2+y^2}\right)=0$ ✓

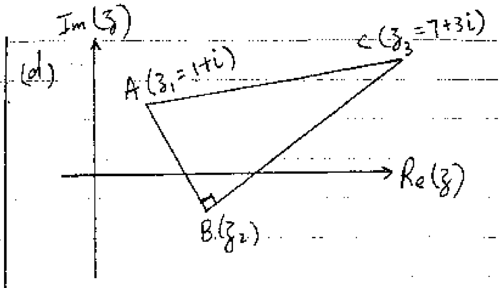
$x-\frac{x}{x^2+y^2}=0$

$\therefore x^2+y^2=1$ ✓

or $x=0$ ✓



(d)



$z_2 = x+iy$

$i\vec{BC} = \vec{BA}$

$i[(7-x)+(3-y)i] = (1-x)+i(1-y)$

$-(3-y)+i(7-x) = (1-x)+i(1-y)$

Equating real part,

$-3+y = 1-x$

$x+y=4$ —— (1)

Equating imag part,

$7-x = 1-y$

$-x+y=-6$ —— (2)

$\left.\right\}$ ✓

solving (1) + (2) simult.

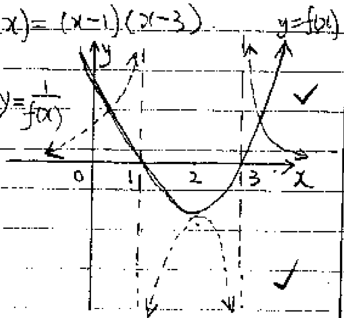
$x=5,\ y=-1$

$z_2 = 5-i$ ✓

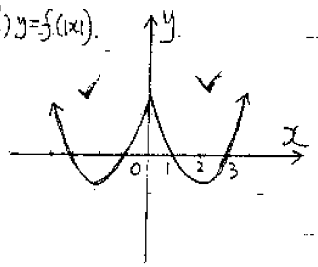
Question 3

(a) $f(x) = (x-1)(x-3)$

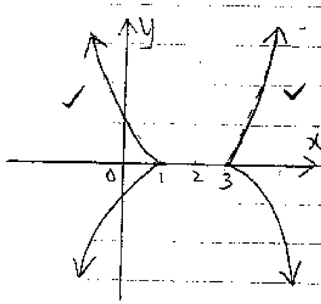
(i) $y = \frac{1}{f(x)}$



(ii) $y = f(|x|)$



(iii) $|y| = f(x)$



(b)(i) $y = \frac{(x+1)^4}{x^4+1}$

$\frac{dy}{dx} = \frac{(x^4+1)4(x+1)^3 - (x+1)^4 4x^3}{(x^4+1)^2}$

$= \frac{4(x+1)^3(1-x^3)}{(x^4+1)^2}$

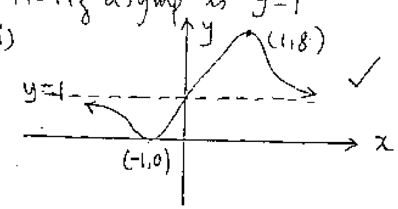
$y' = 0 \Rightarrow x = -1, 1$

Stationary points are
$(-1, 0)$, $(1, 8)$

As $x \to \infty$, $y \to 1$

Horiz. asymp is $y = 1$

(ii)

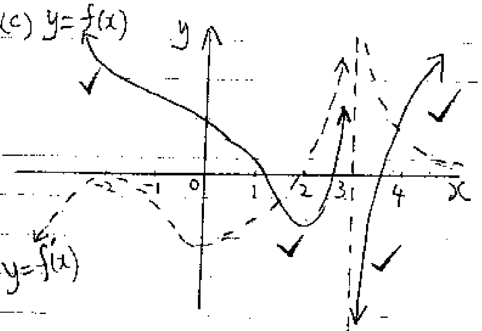


(iii) $(x+1)^4 = k(x^4+1)$ has two distinct real roots

$\therefore$ the line $y = k$ must meet the curve at two points

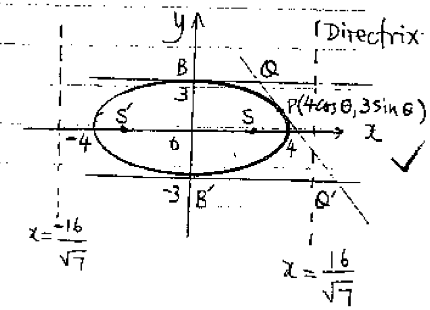
$0 < k < 1$ and $1 < k < 8$

(c) $y = f(x)$



Question 4

(a) (i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$



$b^2 = a^2(1-e^2)$

$9 = 16(1-e^2)$

$e = \frac{\sqrt{7}}{4}$

$S(ae, 0) = (\sqrt{7}, 0)$

$S'(-\sqrt{7}, 0)$

Directrix $x = \pm\frac{a}{e}$

$= \pm 16/\sqrt{7}$

(ii) $\frac{dy}{dx} = -\frac{9x}{16y}$

$= \frac{-3\cot\theta}{4}$

Equation of the tangent at P

$y - 3\sin\theta = \frac{-3\cot\theta}{4}(x - 4\cos\theta)$

$\frac{y\sin\theta}{3} + \frac{x\cos\theta}{4} = 1$

(iii) At Q, $y = 3$

$\therefore \frac{3\sin\theta}{3} + \frac{x\cos\theta}{4} = 1$

$\therefore x_Q = \frac{4(1-\sin\theta)}{\cos\theta}$

At Q', $y = -3$

$\therefore \frac{-3\sin\theta}{3} + \frac{x\cos\theta}{4} = 1$

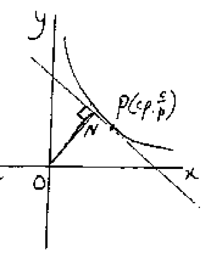
$x_{Q'} = \frac{4(1+\sin\theta)}{\cos\theta}$

$BQ \times B'Q'$

$= \frac{4(1-\sin\theta)}{\cos\theta} \times \frac{4(1+\sin\theta)}{\cos\theta} = 16$

(b) $y = \frac{c^2}{x}$

$y' = \frac{-c^2}{x^2}$



Eqn of tangent at P,

$y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$x + p^2 y = 2cp$ —(1)

Equation of ON is

$y = p^2 x$ —(2)

Solving (1), (2) simult to find coord N

$x_N = \frac{2cp}{1+p^4}$

$y_N = \frac{2cp^3}{1+p^4}$

using $x(1+p^4) = 2cp$ and $\frac{y}{x} = p^2$,

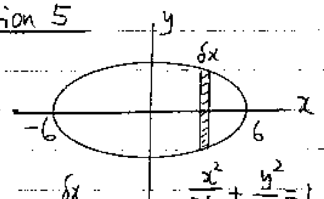
$x^2(1+p^4)^2 = (2cp)^2$

$x^2\left(1+\frac{y^2}{x^2}\right)^2 = 4c^2\left(\frac{y}{x}\right)$

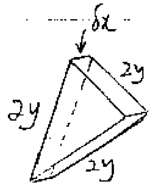
$(x^2+y^2)^2 = 4c^2xy$

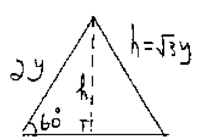
Question 5

(a)



$\frac{x^2}{36}+\frac{y^2}{16}=1$





cross-sectional area of $\Delta = \sqrt{3}y^2$

$\delta V = A\delta x$

$= \sqrt{3}y^2\delta x$ ✓

Vol of the solid

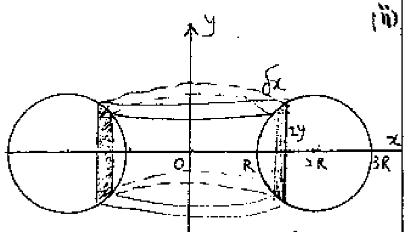
$= \int \sqrt{3}\,y^2 dx$

$= \sqrt{3}\int_{-6}^{6} 16\left(1-\frac{x^2}{36}\right)dx$ ✓

$= 16\sqrt{3}\left[\left(x-\frac{x^3}{108}\right)\right]_{-6}^{6}$ ✓

$= 128\sqrt{3}$ units^3 ✓

(b)



$\delta V = 2\pi x \cdot 2y\,\delta x$

$= 4\pi xy\,\delta x$ ✓

Volume $= \int_R^{3R} 4\pi x\sqrt{R^2-(x-2R)^2}\,dx$ ✓

Let $x-2R = R\sin\theta$

$dx = R\cos\theta\,d\theta$

volume of the torus

$= 4\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(R^2-R^2\sin^2\theta)}\,.\,(2R+R\sin\theta)\ R\cos\theta\,d\theta$ ✓

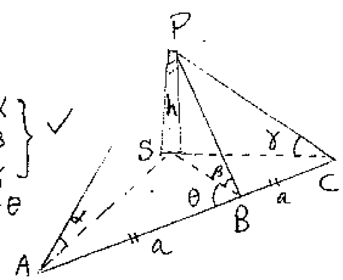
$= 4\pi R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2\cos^2\theta+\cos^2\theta\sin\theta)\,d\theta$

$= 4\pi R^3\left[(\theta+\sin 2\theta)-\frac{1}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ ✓ ✓

$= 4\pi^2R^3$ ✓

(c)

(i) $AS = h\cot\alpha$, $BS = h\cot\beta$, $CS = h\cot\gamma$ ✓



$\angle SBC = 180-\theta$

$\angle ABS = \theta$

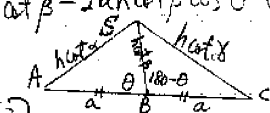
(ii) In ΔBSC,

$(SC)^2 = a^2+h^2\cot^2\beta - 2ah\cot\beta\cos(180-\theta)$,

$h^2\cot^2\gamma = a^2+h^2\cot^2\beta+2ah\cot\beta\cos\theta$ — ① ✓

In ΔABS,

$h^2\cot^2\alpha = a^2+h^2\cot^2\beta-2ah\cot\beta\cos\theta$ — ②



Adding, ① + ②

$h^2(\cot^2\alpha+\cot^2\gamma) = 2a^2+2h^2\cot^2\beta$ ✓

$h^2(\cot^2\alpha+\cot^2\gamma-2\cot^2\beta) = 2a^2$

$h = \frac{a\sqrt{2}}{[\cot^2\alpha+\cot^2\gamma-2\cot^2\beta]^{\frac{1}{2}}}$ ✓

Question 6

(a) a, b, c are roots of $x^3+qx+r=0$

Let $y = b+c-2a$

$= (b+c+a)-3a$

$= 0-3a$

$\therefore a = -\frac{y}{3}$ ✓

$\left(\frac{-y}{3}\right)^3+q\left(\frac{-y}{3}\right)+r=0$ ✓

$y^3+9qy-27r=0$ ✓

(b) (i) $\tan 3\theta = \tan(2\theta+\theta)$

RHS $= \frac{\tan 2\theta+\tan\theta}{1-\tan 2\theta\tan\theta}$ ✓

$= \frac{\frac{2\tan\theta}{1-\tan^2\theta}+\tan\theta}{1-\frac{2\tan\theta}{1-\tan^2\theta}\tan\theta}$

$= \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$ ✓

(ii) let $\theta = \tan^{-1}x$

$\therefore \tan(3\tan^{-1}x) = \frac{3x-x^3}{1-3x^2}$

using $3\tan^{-1}x = \frac{\pi}{2}-\tan^{-1}(3x)$

$\tan(3\tan^{-1}x) = \tan\left(\frac{\pi}{2}-\tan^{-1}(3x)\right)$ ✓

$= \cot(\tan^{-1}3x)$

$= \frac{1}{\tan[\tan^{-1}(3x)]}$ ✓

$\frac{3x-x^3}{1-3x^2} = \frac{1}{3x}$

$3x^4-12x^2+1=0$ ✓

$x^2 = \frac{12\pm\sqrt{144-12}}{6}$

$\therefore x = 0.292$ only ✓

$x = 1.979$ is not the solut.

(c) (i) $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\cdots\cdots$

$+\frac{1}{(n+1)(n+2)(n+3)}$

$= \frac{(n+1)(n+4)}{4(n+2)(n+3)}$

When $n=0$, LHS $=\frac{1}{6}$, RHS $=\frac{1}{6}$ ✓

Assume that it is true for $n=K$

$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\cdots+\frac{1}{(K+1)(K+2)(K+3)}$

$= \frac{(K+1)(K+4)}{4(K+2)(K+3)}$ ✓

RTP: $S_{K+1} = S_K+T_{K+1}$

$= \frac{(K+2)(K+5)}{4(K+3)(K+4)}$

$S_{K+1} = \frac{1}{(K+2)(K+3)(K+4)}+\frac{(K+1)(K+4)}{4(K+2)(K+3)}$

$= \frac{4+(K+1)(K+4)(K+4)}{4(K+3)(K+4)(K+2)}$

$= \frac{(K+1)(K^2+8K+16)+4}{4(K+2)(K+3)(K+4)}$

$= \frac{(K+2)^2(K+5)}{4(K+2)(K+3)(K+4)}$

$= \frac{(K+2)(K+5)}{4(K+3)(K+4)}$ ✓

Since it is true for $n=0$,

it is proven true for $n=k+1$.

$\therefore$ it is true for $n=0+1,=1$

true for $n=1+1=2$

$\therefore$ it is true for all $n\geq 0$ ✓

(ii) $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{(r+1)(r+2)(r+3)}$

$= \frac{(n+1)(n+4)}{4(n+2)(n+3)}$

$= \frac{(\frac{n}{n}+\frac{1}{n})(\frac{n}{n}+\frac{4}{n})}{4(\frac{n}{n}+\frac{2}{n})(\frac{n}{n}+\frac{3}{n})}$ ✓

$= \frac{1\times1}{4\times1\times1} = \frac{1}{4}$ ✓

Question 7

(a) (i) $I_n = \int_0^{\pi} \sin^n\theta \, d\theta$

$= \int \sin\theta . \sin^{n-1}\theta \, d\theta$

$= \left[-\cos\theta . \sin^{n-1}\theta\right]_0^{\pi} - \int -\cos\theta . (n-1)\sin^{n-2}\theta . \cos\theta \, d\theta$ ✓

$= 0 + (n-1)\int \cos^2\theta \sin^{n-2}\theta \, d\theta$

$= (n-1)\int_0^{\pi} (1-\sin^2\theta)\sin^{n-2}\theta \, d\theta$

$= (n-1)\int_0^{\pi} \sin^{n-2}\theta \, d\theta - (n-1)\int_0^{\pi} \sin^n\theta \, d\theta$ ✓

$= (n-1) I_{n-2} - (n-1) I_n$

$n I_n = (n-1) I_{n-2}$ ✓

(ii) $I_8 = \frac{7}{8} I_6$

$= \frac{7}{8} \frac{5}{6} I_4$

$= \frac{7}{8} . \frac{5}{6} . \frac{3}{4} . I_2$

$= \frac{7}{8} \frac{5}{6} . \frac{3}{4} \frac{1}{2} I_0$

$= \frac{7}{8} \frac{5}{6} . \frac{3}{4} . \frac{1}{2} \pi$

$= \frac{35\pi}{128}$ ✓

(iii) $\int_0^{\pi} x\sin^n x \, dx = \int_0^{\pi} (\pi - x)\sin^n(\pi - x) \, dx$

$= \int_0^{\pi} (\pi - x)\sin^n x \, dx$

$= \pi\int_0^{\pi} \sin^n x \, dx - \int_0^{\pi} x\sin^n x \, dx$ ✓

$2\int_0^{\pi} x\sin^n x \, dx = \pi\int_0^{\pi} \sin^n x \, dx$

$\therefore \int_0^{\pi} x\sin^n x \, dx = I_n\left(\frac{\pi}{2}\right)$ ✓

(b) $x = \cos\theta + i\sin\theta$

(i) $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

$x^n + \frac{1}{x^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ ✓

(ii) $5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$

$5\left(x^2 + \frac{1}{x^2}\right) - 11\left(x + \frac{1}{x}\right) + 16 = 0$ ✓

$5 . 2\cos 2\theta - 11 . 2\cos\theta + 16 = 0$

$10\cos^2\theta - 11\cos\theta + 3 = 0$ ✓

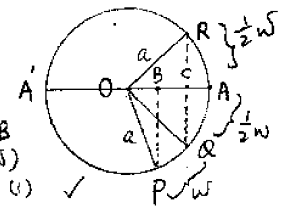
$(5\cos\theta - 3)(2\cos\theta - 1) = 0$

$\cos\theta = \frac{3}{5}$ or $\cos\theta = \frac{1}{2}$

$x = \frac{3}{5} + \frac{4}{5}i$, $x = \frac{3}{5} - \frac{4}{5}i$ ✓

$x = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $x = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ ✓

(c)



OB = 1 unit,
OC = 5 units

$OB = a\cos\angle POB$

$1 = a\cos(\omega + \frac{1}{2}\omega)$

$= a\cos\frac{3\omega}{2}$ —— (1) ✓

$OC = a\cos\angle QOA$

$5 = a\cos\frac{\omega}{2}$ —— (2) ✓

$\frac{(2)}{(1)}$ $5 = \frac{\cos\frac{1}{2}\omega}{\cos\frac{3\omega}{2}}$

$5\cos 3\left(\frac{\omega}{2}\right) = \cos\frac{1}{2}\omega$

$5\left[4\cos^3\left(\frac{1}{2}\omega\right) - 3\cos\frac{1}{2}\omega\right] = \cos\frac{1}{2}\omega$

$\cos^2\left(\frac{1}{2}\omega\right) = \frac{16}{20} \implies \cos^2\left(\frac{\omega}{2}\right) = \frac{4}{5}$

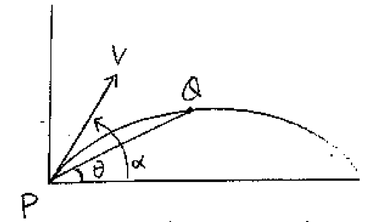
$\cos\omega = 2\cos^2\frac{1}{2}\omega - 1$

$= \frac{3}{5}$ ✓

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$ ✓

Question 8

(a)



Let $\dot{x}$ and $\dot{y}$ be the horizon. and vertical components of velocity when the ball is at Q

$\dot{x} = V\cos\alpha$, $\dot{y} = V\sin\alpha - gt$ ✓

$\tan\theta = \frac{y}{x}$

$= \frac{V\sin\alpha \, t - \frac{1}{2}gt^2}{V\cos\alpha \, t}$ ✓

$\therefore \tan\theta = \tan\alpha - \frac{gt}{2V\cos\alpha}$

The direction of motion at Q is inclined to the horizon. at $\tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) = \tan^{-1}\left[\tan\alpha - \frac{gt}{V\cos\alpha}\right]$ ✓

But $\frac{gt}{V\cos\alpha} = 2\tan\alpha - 2\tan\theta$

$\therefore \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) = \tan^{-1}\left[\tan\alpha - 2\tan\alpha + 2\tan\theta\right]$

$= \tan^{-1}\left[2\tan\theta - \tan\alpha\right]$ ✓

(b) (i) $x = V\cos\theta \; t$

$y = V\sin\theta . t - \frac{1}{2}gt^2$

Time of flight $\implies y = 0$

$\implies t = \frac{2V\sin\theta}{g}$ ✓

$\therefore$ Range $= V\cos\theta \times \frac{2V\sin\theta}{g}$

$R = \frac{V^2\sin 2\theta}{g}$ ✓

(ii) $R - p = \frac{V^2\sin 2\alpha}{g}$ —— (1) ✓

$R + q = \frac{V^2\sin 2\beta}{g}$ —— (2) ✓

$\frac{V^2\sin 2\theta}{g} - p = \frac{V^2\sin 2\alpha}{g}$

$\frac{V^2\sin 2\theta}{g} + q = \frac{V^2\sin 2\beta}{g}$

$\frac{p}{q} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$ ✓

$(p+q)\sin 2\theta = p\sin 2\beta + q\sin 2\alpha$

$\sin 2\theta = \frac{p\sin 2\beta + q\sin 2\alpha}{p+q}$ ✓

(c) (i) $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ ✓

(ii) ${}^3C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$

$= \frac{27}{64}$ ✓

(iii) ${}^3C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + {}^3C_3\left(\frac{3}{4}\right)^3$

$= \frac{27}{32}$ ✓

(iv) ${}^3C_2\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)$ ✓

$= \frac{3}{32}$ ✓

— END OF PAPER —