



Barker College

**2001
TRIAL
HIGHER SCHOOL
CERTIFICATE**

Mathematics Extension 2

AM WEDNESDAY 8 AUGUST

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 10.
- ALL necessary working should be shown in every question

Total marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value

Total marks (120)

Attempt Questions 1 – 8

ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 [15 marks] [START A NEW PAGE]

(a) Find

$$\int \frac{dx}{x^2 - 16x + 80}$$

2

(b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin \theta} d\theta$

3

(ii) $\int_0^2 \frac{8 dx}{(x + 2)(x^2 + 4)}$

4

(iii) $\int_0^{\pi} e^x \cos x dx$

3

(c) Find $\int \frac{2x}{\sqrt{4x - x^2}} dx$

You may wish to use the substitution of $u = x - 2$.

3

Question 2 [15 marks] [START A NEW PAGE]

(a) (i) Find all real numbers x and y such that $(x + iy)^2 = -3 + 4i$. 2

(ii) Hence, solve the equation $z^2 - 3z + (3 - i) = 0$. 1

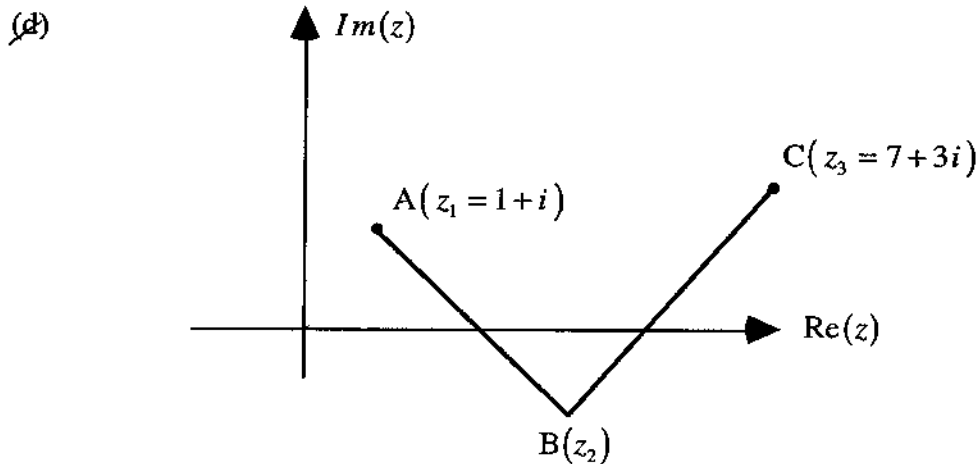
(b) (i) Express $\sqrt{3} + i$ and $\sqrt{3} - i$ in modulus-argument form. 2

(ii) Hence, simplify $(\sqrt{3} + i)^{15} + (\sqrt{3} - i)^{15}$ 1

(c) Sketch the locus specified by

(i) $|z| \leq |z - 2|$ and $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$ 3

(ii) $\operatorname{Re}\left(z - \frac{1}{z}\right) = 0$ (State the equation(s) of the locus). 4



The points A and C represent the complex numbers

$$z_1 = 1 + i \text{ and } z_3 = 7 + 3i$$

Find the complex number z_2 represented by B such that $\triangle ABC$ is isosceles and right angled at B.

2

Question 3 [15 marks] [START A NEW PAGE]

(a) If $f(x) = (x - 1)(x - 3)$ then sketch

(i) $y = \frac{1}{f(x)}$ 2

(ii) $y = f(|x|)$ 2

(iii) $|y| = f(x)$ 2

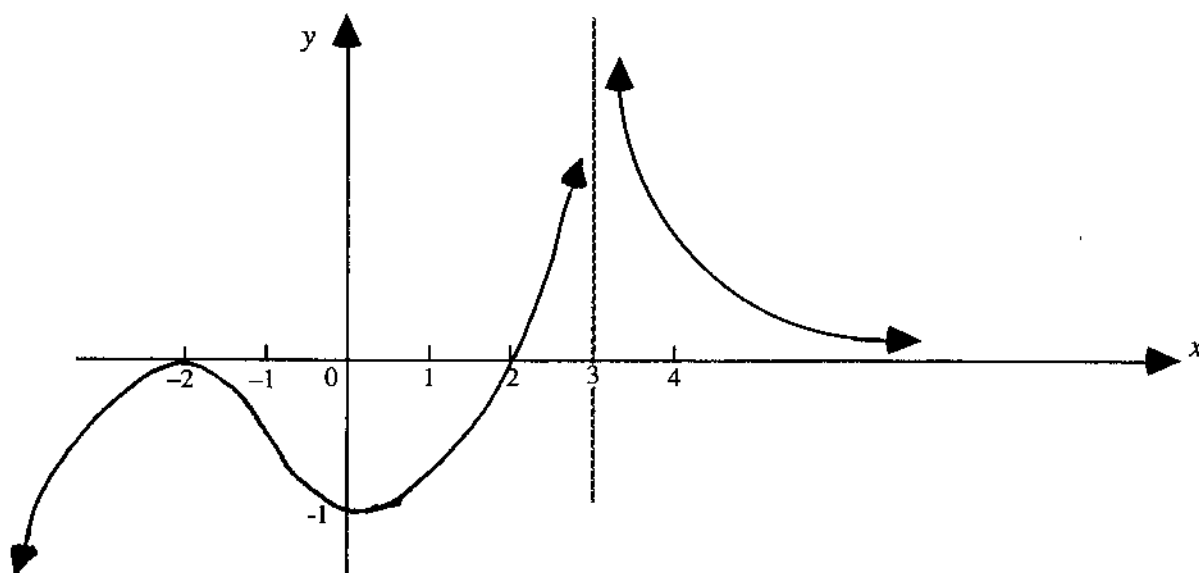
(b) (i) Find the stationary points and the asymptotes of the function

$$y = \frac{(x + 1)^4}{x^4 + 1}$$
 2

(ii) Sketch this function labelling all essential features. 1

(iii) Use the graph to find the set of values of k for which $(x + 1)^4 = k(x^4 + 1)$ has two distinct real roots. 2

(c) Given the graph of $y = f'(x)$ below, sketch the graph of $y = f(x)$.
 $y = f'(x)$ is the derivative of $y = f(x)$. 4



Question 4 [15 marks] [START A NEW PAGE]

(a) An ellipse has the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$

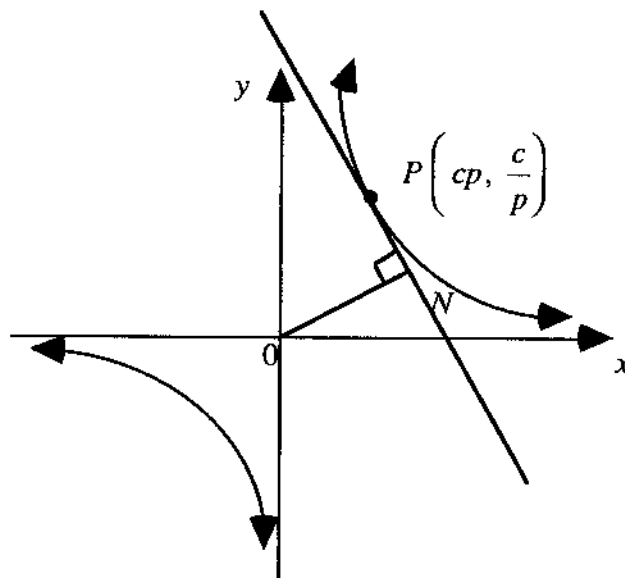
(i) Sketch the ellipse showing the foci S and S' and the directrices. 4

(ii) Prove that the tangent at the point $P(4\cos\theta, 3\sin\theta)$ to the ellipse has the equation $\frac{x\cos\theta}{4} + \frac{y\sin\theta}{3} = 1$ 3

(iii) The ellipse meets the y -axis at B and B' . The tangents at B and B' meet the tangent at P at the points Q and Q' .
Prove that $BQ \cdot B'Q' = 16$ 3

(b) The line through O perpendicular to the tangent at $P\left(cp, \frac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ meets the tangent at N .

Find the coordinates of N and show that as p varies, the locus of N is $(x^2 + y^2)^2 = 4c^2xy$. 5



Question 5 [15 marks] [START A NEW PAGE]

(a) A solid has as its base the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$

If each section perpendicular to the major axis is an equilateral triangle, show that the volume of the solid is $128\sqrt{3}$ units³.

4

(b) The region $(x - 2R)^2 + y^2 \leq R^2$ is rotated about the y-axis forming a solid of revolution called a torus.

By summing volumes of cylindrical shells, show that the volume of the torus is $4\pi^2 R^3$ units³.

6

(c) The angles of elevation of the top of a tower P measured from three points A, B, C are α , β , γ respectively. A, B, C are in a straight line such that $AB = BC = a$, but the line AC does not pass through S, the base of the tower.

(i) If $\angle ABS = \theta$, show that

$$(CS)^2 = a^2 + h^2 \cot^2 \beta + 2ah \cot \beta \cos \theta$$

2

(ii) Prove that the height of the tower is

$$\frac{a\sqrt{2}}{\{\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta\}^{\frac{1}{2}}}$$

3

Question 6 [15 marks] [START A NEW PAGE]

- (a) Given that a, b and c are the roots of the equation $x^3 + qx + r = 0$, find the cubic equation in y , in terms of q and r , whose roots are $(b + c - 2a), (c + a - 2b)$ and $(a + b - 2c)$

3

- (b) Using $\tan 3\theta = \tan(2\theta + \theta)$, show that

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

2

- (ii) Find the value of x for which $3 \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}(3x)$ where $\tan^{-1} x$ and $\tan^{-1}(3x)$ both lie between 0 and $\frac{\pi}{2}$

4

- (c) Using mathematical induction, prove that

$$\sum_{r=0}^n \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+1)(n+4)}{4(n+2)(n+3)}$$

4

- (iii) Hence, find $\lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{(r+1)(r+2)(r+3)}$

2

Question 7 [15 marks] [START A NEW PAGE]

(a) ~~(i)~~ Given that $\int_0^\pi \sin^n \theta d\theta = I_n$, prove that

$$nI_n = (n - 1)I_{n-2}$$

3

~~(ii)~~ Hence, evaluate I_8

1

~~(iii)~~ Use the result $\int_0^a f(x)dx = \int_0^a f(a - x)dx$ to show that

$$\int_0^\pi x \sin^n x dx = \left(\frac{\pi}{2}\right) I_n$$

2

(b) ~~(i)~~ Use De Moivre's Theorem to prove that, if $2\cos\theta = x + \frac{1}{x}$,
then $2\cos n\theta = x^n + \frac{1}{x^n}$

1

~~(ii)~~ Hence, or otherwise, solve the equation

$$5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$$

4

(c) At the ends of three successive seconds, the distances of a point moving with Simple Harmonic Motion from its mean position, measured in the same direction, are 1, 5 and 5 metres.

Show that the period of the complete oscillation is $\frac{2\pi}{\cos^{-1}\left(\frac{3}{5}\right)}$ seconds.

4

Question 8 [15 marks] [START A NEW PAGE]

- (a) A ball thrown from a point P with velocity V , at an inclination α to the horizontal reaches a point Q after t seconds.
Show that if PQ is inclined at θ to the horizontal, (where $\alpha > \theta$), then the direction of motion of the ball, when at Q , is inclined to the horizontal at an acute angle of $\tan^{-1}[2 \tan \theta - \tan \alpha]$.

You may use the result without proof

$$x = V \cos \alpha \times t$$

$$y = V \sin \alpha \times t - \frac{1}{2} g t^2$$

4

- (b) (i) A gun fires shells with muzzle velocity V . Ignoring air resistance, show that the range on a horizontal plane is $\frac{V^2 \sin 2\theta}{g}$ where θ is the angle of elevation of the gun and g is the acceleration due to gravity.

2

- (ii) The gun and the target lie on the same horizontal plane. The gun fires, in the correct vertical plane, at the target using an angle of elevation α and the shell falls short by a distance p . When the angle of elevation is changed to β , the shell overshoots the target by a distance q .

$$\text{Show that } \sin 2\theta = \frac{p \sin 2\beta + q \sin 2\alpha}{p + q}$$

4

- (c) Fred has three uniform tetrahedra (triangular pyramids). Each of these tetrahedra has one face black, one face white, one face red and one face green.
When tossed onto a table, three faces of each tetrahedron can be seen.
If the probability of any coloured face not being seen is equally likely, what is the probability that

- (i) no black face can be seen?

1

- (ii) exactly 2 black faces can be seen?

1

- (iii) at least 2 red faces can be seen?

1

- (iv) 3 white faces and only 1 green face can be seen?

2

END OF PAPER