ASCHAM SCHOOL

EXTENSION 2 MATHEMATICS

2001 TRIAL EXAMINATION

Time : 3 hours + 5 minutes reading time

Instructions:

Attempt all questions

All questions are of equal value

All necessary working should be shown for every question.

Full marks may not be awarded for careless or badly arranged work

A Table of Standard Integrals is provided

Approved calculators may be used

Each question should be answered in a separate booklet

Question 1

(a) T $(a\cos\theta, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre O. A line through O, parallel to the tangent at T, meets the ellipse at M and N.

(i) Show the gradient of the tangent at T is $-\frac{b\cos\theta}{a\sin\theta}$ and find the equation of MN. [3]

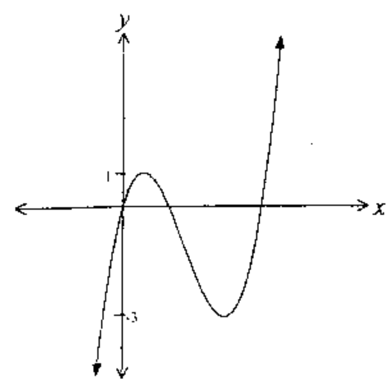
(ii) Show that M and N are $(-a\sin\theta, b\cos\theta)$ and $(a\sin\theta, -b\cos\theta)$ [3]

(iii) Show that the area of ΔTMN is independent of θ. [5]

(b) Describe the locus $|z-3|+|z+3|=10$ [4]

Question 2

(a)



$y=f(x)$ is drawn above. Copy the diagram into your answer booklet and on the same diagram sketch $y=\log_e f(x)$. [2]

(b) Find the volume of the solid formed when the arc of $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the line $y = 2$ [6]

(c) A dome has a circular base of radius 10 metres. Cross-sections perpendicular to the base and one axis are parabolas whose height is the same as the base width.

(i) Why would Simpson's rule give the exact area of the parabolic cross-section? [1]

(ii) Show that the area of the parabolic cross-section is $\frac{8y^2}{3}$ square metres. [3]

(iii) Find the volume of the dome. [3]

Question 3

(a) (i) Express $-1+i$ in modulus argument form [1]

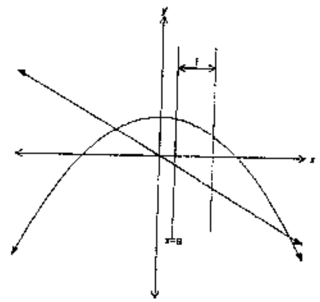
(ii) Hence evaluate $(-1+i)^{10}$ [2]

(b) (i) Find all pairs of integers x and y such that $(x+iy)^2 = -3-4i$ [2]

(ii) Hence or otherwise, solve the quadratic equation $z^2 - 3z + (3+i) = 0$ [2]

(c) Show, by geometrical means or otherwise that, if z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$, then $\frac{z_1 + z_2}{z_1 - z_2}$ is pure imaginary. [3]

(d) Consider the area between the curves $y = 3 - x^2$ and $y = -2x$. Suppose that two vertical lines 1 unit apart cross this area.



(i) If the first line is x = a, write an expression for the shaded area. [3]

(ii) Find the maximum value of the shaded area. [2]

Question 4

(a) Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}}dx$ [3]

(b) Find the exact value of (i) $\int_1^e \log_e x \, dx$ [2]

(ii) $\int_0^{\ln 3} e^x \operatorname{cosec}^2(e^x) dx$ [3]

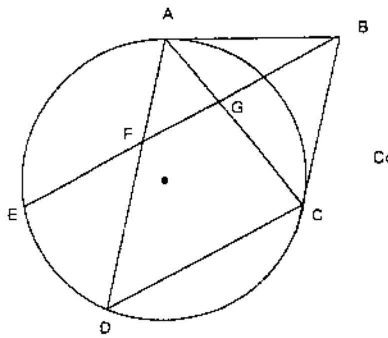
(c) (i) Using the substitution $u = \frac{1}{x}$, show that $\int_{0.5}^{1} \frac{\ln x}{1+x^2} dx = \int_{2}^{1} \frac{\ln u}{1+u^2} du$ [2]

(ii) Deduce the value of $\int_{0.5}^{2} \frac{\ln x}{1+x^2} dx$ [2]

(d) Find $\int \frac{\cos x}{\sin x + \sin^2 x} dx$ [3]

Question 5

(a)



Copy the diagram into your examination booklet

In the diagram EB is parallel to DC. Tangents from B meet the circle at A and C. Prove that

(i) $\angle BCA = \angle BFA$ [3]

(ii) ABCF is a cyclic quadrilateral [1]

(iii) DF = CF [3]

(b) (i) Draw the graph of $y = \dfrac{x^4 - 1}{x^2}$ [2]

(ii) On separate axes sketch $y = \tan^{-1}\left(\dfrac{x^4 - 1}{x^2}\right)$ [2]

(c) (i) On the same axes sketch $y = |x| - 2$ and $y = 4 + 3x - x^2$ [2]

(ii) Hence or otherwise solve $\dfrac{|x| - 2}{4 + 3x - x^2} > 0$ [2]

Question 6

(a) Graph the intersection of:

$z\bar{z} \geq 9 \quad z + \bar{z} \leq 8 \quad 0 < Arg(z) < \frac{\pi}{4}$ [4]

(b) Let α be the complex root of the polynomial $z^7 = 1$ with the smallest possible argument.

Let $\theta = \alpha + \alpha^2 + \alpha^4$ and

$\delta = \alpha^3 + \alpha^5 + \alpha^6$

(i) Explain why $\alpha^7 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = 0$ [1]

(ii) Show that $\theta + \delta = -1$ and $\theta\delta = 2$ and hence write a quadratic equation whose roots are θ and δ [3]

(iii) Show that $\theta = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $\delta = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$ [2]

(iv) Write down α in modulus-argument form, and show that

$\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$ and $\sin\frac{4\pi}{7} - \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$ [5]

Question 7

(a) The roots of a cubic equation are α, β and γ, and $\sum \alpha^n = \alpha^n + \beta^n + \gamma^n$
It is given that $\sum \alpha = -1$, $\sum \alpha^2 = 7$, $\sum \alpha^3 = 8$

(i) Deduce that the equation is $x^3 + x^2 - 3x - 6 = 0$ [2]

(ii) Hence evaluate $\sum \alpha^4$ [2]

(b) (i) If $I_n = \int x(\ln x)^n dx$ for $n \geq 0$, show that $I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}$ [2]

(ii) Hence, find $\int x(\ln x)^2 dx$ [2]

(c) A particle is projected from the origin at an angle of $\alpha°$ with initial velocity V, and it passes through a point (m,n).

(i) Prove that $gm^2 \tan^2 \alpha - 2mV^2 \tan \alpha + gm^2 + 2nV^2 = 0$
where g is acceleration due to gravity [4]

(ii) Prove that there are two possible trajectories if

$$\left(V^2 - gn\right)^2 > g^2\left(m^2 + n^2\right)$$

[3]

Question 8

(a) A chord AB and a diameter CD, of a circle centre O, intersect at M within the circle. M is not the centre.

(i) Show that $(CM + MD)^2 > (AM + MB)^2$ [2]

(ii) Deduce that $(CM - MD)^2 > (AM - MB)^2$ [2]

(b) A particle of mass *m* kg falls from rest in a medium where the resistance to motion is *mkv* when the particle has velocity *v* m/s.

(i) Draw a diagram showing the forces acting on the particle. [1]

(ii) Show that the equation of motion of the particle is $\ddot{x} = k(V - v)$ where V m/s is the terminal velocity of the particle in this medium, and x metres is the distance fallen in t seconds. [2]

(iii) Find in terms of V and k the time T seconds taken for the particle to attain 50% of its terminal velocity, and the distance fallen in this time. [4]

(iv) What percentage if its terminal velocity will the particle have attained in time $2T$ seconds? Sketch a graph of v against t showing this information. [3]

(v) If the particle has reached 87.5% of its terminal velocity in 15 seconds, find the value of k. [1]

End of Examination